m^{th} POWER SYMMETRIC n-SIGRAPHS

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Abstract. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. A *symmetric n-sigraph* (*symmetric n-marked graph*) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. The m^{th} *power graph* of a graph $G = (V, E)$ is a graph $G^m = (V, E')$, with same vertex set as G, and has two vertices u and v are adjacent if their distance in G is m or less. Analogously, one can define the m^{th} *power symmetric n-sigraph* S_n^m of a symmetric n-sigraph $S_n = (G, \sigma)$ as a symmetric n-sigraph, $S_n^m = (G^m, \sigma')$, where G^m is the underlying graph of S_n^m, and for any edge $e = uv$ in S_n^m, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$,

$$\mu(v) = \prod_{u \in N(v)} \sigma(uv).$$

It is shown that for any symmetric n-sigraph S_n, its m^{th} power symmetric n-sigraph S_n^m is i-balanced. We then give structural characterization of m^{th} power symmetric n-sigraphs. Further, we obtain some switching equivalence relationship between m^{th} power symmetric n-sigraph and line symmetric n-sigraph.

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1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric* if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph* (*symmetric n-marked graph*) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [6], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (see, also, R. Rangarajan and P. Siva Kota Reddy [3]):

Definition. Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [6].

Proposition 1 (**E. Sampathkumar et al. [6]**) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [6], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows:

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by

$\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [6]).

Proposition 2 (**E. Sampathkumar et al. [6]**) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. m^{th} Power n-sigraph

The *m^{th} power graph G^m* of a graph G is defined in [1] as follows: The m^{th} power graph has same vertex set as G, and has two vertices u and v are adjacent if their distance in G is n or less.

In this paper, we introduce a natural extension of the notion of m^{th} power graphs to the realm of n-sigraphs: Consider the n-marking μ on vertices of S_n defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. The *m^{th} power n-sigraph* of S_n is an n-sigraph $S_n^m = (G^m, \sigma')$, where G^m is the underlying graph of S^m, where for any edge $e = uv \in G^m$, $\sigma'(uv) = \mu(u)\mu(v)$.

Hence, we shall call a given n-sigraph $S_n = (G, \sigma)$ a *m^{th} power n-sigraph* if it is isomorphic to the m^{th} power sigraph $(S'_n)^m = ((G')^m, \sigma')$ of some n-sigraph S'_n.

In the following subsection, we shall present a characterization n-sigraphs which are m^{th} power n-sigraphs.

2.1. Switching invariant m^{th} power n-sigraphs

The following result indicates the limitations of the notion of m^{th} power n-sigraphs as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to m^{th} power n-sigraphs.

Proposition 3 *For any n-sigraph $S_n = (G, \sigma)$, its m^{th} power n-sigraph S_n^m is i-balanced.*

Proof. Let σ' denote the labelling of S_n^m. Then by definition of S_n^m, we see that $\sigma'(uv) = \mu(u)\mu(v)$, for every edge uv of S_n^m and hence, by Proposition 1, the result follows. ∎

The following result characterizes n-sigraphs which are m^{th} power n-sigraphs.

Proposition 4 *An n-sigraph $S_n = (G, \sigma)$ is a m^{th} power n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a m^{th} power graph.*

Proof. Suppose that S_n is i-balanced and G is a m^{th} power graph. Then there exists a graph H such that $H^m \cong G$. Since S_n is i-balanced, by Proposition 1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $(S'_n)^m \cong S_n$. Hence S_n is a m^{th} power n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a m^{th} power n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $(S'_n)^m \cong S_n$. Hence G is the m^{th} power graph of H and by Proposition 3, S_n is i-balanced. ∎

For any positive integer k, the k^{th} iterated m^{th} power n-sigraph, $(S_n^m)^k$ of S_n is defined as follows:

$$(S_n^m)^0 = S_n, \ (S_n^m)^k = S_n^m((S_n^m)^{k-1}).$$

Corollary 3.1. *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(S_n^m)^k$ is i-balanced.*

The *line n-sigraph* $L(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is defined as follows (See [7]): $L(S_n) = (L(G), \sigma')$, where for any edge ee'in $L(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$.

Proposition 5 (**E. Sampathkumar et al.** [**7**]) *For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n)$ is i-balanced.*

For any positive integer k, the k^{th} iterated line n-sigraph, $L^k(S_n)$ of S_n is defined as follows:

$$L^0(S_n) = S_n, \ L^k(S_n) = L(L^{k-1}(S_n)).$$

Corollary 5.1. *For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $L^k(S_n)$ is i-balanced.*

3. Switching equivalence of line n-sigraphs and m^{th} power n-sigraphs

We now characterize n-sigraphs whose line n-sigraphs and its m^{th} power n-sigraphs are switching equivalent. In the case of graphs the following result is due to J. Akiyama et al. [1].

Proposition 6 (**J. Akiyama et al.** [**1**]) *For any $m \geq 2$, the solutions to the equation $L(G) \cong G^m$ are graphs $G = pK_3$, where p is an arbitrary integer.*

Proposition 7 *For any n-sigraph $S_n = (G, \sigma)$, $L(S_n) \sim S_n^m$, where $m \geq 2$ if, and only if, G is pK_3, where p is an arbitrary integer.*

Proof. Suppose $L(S_n) \sim S_n^m$. This implies, $L(G) \cong G^m$ and hence by Proposition-18, we see that the graph G must be isomorphic to pK_3.

Conversely, suppose that G is mK_3. Then $L(G) \cong G^m$ by Proposition-18. Now, if S_n is an n-sigraph with underlying graph as pK_3, by Proposition-5 and 3, $L(S_n)$ and S_n^m are i-balanced and hence, the result follows from Proposition-2. ∎

Note that for $m = 1$, in the above Proposition is reduced to the following result of R. Rangarajan et al. [4].

Proposition 8 (**R. Rangarajan et al.** [**4**]) *For any n-sigraph $S_n = (G, \sigma)$, $L(S_n) \sim S_n$ if, and only if, S_n is an i-balanced n-sigraph which is 2-regular.*

4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^t = at$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an n-sigraph $S_n = (G, \sigma)$, written $(S_n^m)^t$, is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^t.

For an n-sigraph $S_n = (G, \sigma)$, the S_n^m is i-balanced (Proposition 3) and $L(S_n)$ is also i-balanced (Proposition 5). We now examine, the conditions under which t-complement of S_n^m and $L(S_n)$ are i-balanced, where for any $t \in H_n$.

Proposition 9 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $t \in H_n$,*

(i) *If G^m is bipartite then $(S_n^m)^t$ is i-balanced.*

(ii) *If $L(G)$ is bipartite then $(L(S_n))^t$ is i-balanced.*

Proof. (i) Since, by Proposition 3, S_n^m is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in S_n^m whose k^{th} co-ordinate are $-$ is even. Also, since G^m is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in S_n^m whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any t-complement, where for any $t \in H_n$. Hence $(S_n^m)^t$ is i-balanced.

Similarly (ii) follows. ∎

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