AN ANALYTICAL SOLUTION OF FLUID FLOW THROUGH NARROWING SYSTEMS

A.D. Patel
I.A. Salehbhai
Department of Mathematics
S.V. National Institute of Technology
Surat, Gujarat
India

J. Banerjee
Department of Mechanical Engineering
S.V. National Institute of Technology
Surat, Gujarat
India

V.K. Katiyar
Department of Mathematics
Indian Institute of Technology
Roorkee
India

A.K. Shukla
Department of Mathematics
S.V. National Institute of Technology
Surat, Gujarat
India
e-mail: ajayshukla2@rediffmail.com

Abstract. Narrowing of pipeline network is an important aspect in drinking water distribution systems, sewage system and in oil-well techniques. In the proposed problem, a flow equation in simple pipeline network has been studied to solve the velocity flow. The deposition causing narrowing has been replaced by using sinusoidal model with axial velocity. In this paper, we used MAPLE 11.02 for plotting the graphs.

Keywords: narrowing systems, finite Hankel transforms, Laplace transforms, Bessel functions.
AMS Classification: 76D05, 65R10, 44A10, 44A15, 33C10.

Nomenclature

- D Substantial derivative,
- V Velocity vector,
- t Time,
- ρ Density,
- p Pressure,
- μ Dynamic viscosity,
- ν Kinematic viscosity

1. Introduction and preliminaries

The continuity and Navier-Stokes equations (Murlidhar and Biswas [1]) for incompressible flow are:

$$\nabla \cdot V = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{1.1}$$

$$\rho\left(\frac{DV}{Dt}\right) = -\nabla p + \mu \nabla^2 V, \tag{1.2}$$

Equation (1.2) can be easily reduce to r, θ and z directions follows as

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)\end{aligned} \tag{1.3}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} \\ = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\mu}{\rho}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right)\end{aligned} \tag{1.4}$$

$$\begin{aligned}\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} \\ = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\end{aligned} \tag{1.5}$$

This equation plays a very important role in mathematical modeling of real world problems and can also be reduce to different form of equation by applying specific conditions. In the present paper, our aim is to construct a mathematical model for the study of fluid flow in narrowing systems by using (1.2) with the help of equation of continuity and applying the Laplace and finite Hankel transform techniques, which yields the analytical solution.

Several bio-mathematicians (Verma et. al [2], [3], Ponalagusamy [4], Chaturani et. al. [5], [6], [7]) applied the concept of the narrowing system in the study of blood flow through a stenosised artery by using different mathematical tools.

2. Used integral transforms and special functions

The Laplace Transform (Debnath [8]) is defined as,

$$L\{f(x)\} = \int_0^\infty e^{-st} f(t)\, dt \tag{2.6}$$

The zero order Bessel function $J_0(x)$ (Rainville [9]) is defined as

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m} \tag{2.7}$$

The zero order finite Hankel transform (Debnath [8]) is defined as,

$$H_0\left\{f\left(r\right)\right\} = \tilde{f}_0\left(\lambda_n\right) = \int_0^R r\,f\left(r\right)J_0\left(r\lambda_n\right)dr, \tag{2.8}$$

where λ_n are the roots of the equation $J_0\left(R\lambda_n\right) = 0$.

In 1903, Mittag-Leffler [10] introduced the function $E_\alpha\left(z\right)$, defined as

$$E_\alpha\left(z\right) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha n + 1\right)}, \tag{2.9}$$

where z is a complex variable and $\Gamma\left(s\right)$ is a gamma function, $\alpha \geq 0$.

In 1905, Wiman [11] introduced the generalization of $E_\alpha\left(z\right)$ as

$$E_{\alpha,\beta}\left(z\right) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha n+\beta\right)} = \frac{1}{2\alpha\pi\,i}\int\frac{e^{\xi^{\frac{1}{\alpha}}}\xi^{\frac{1-\beta}{\alpha}}}{\xi - z}d\xi, \tag{2.10}$$
$$\left(\alpha,\beta \in \mathrm{C};\ Re\left(\alpha\right) > 0,\ Re\left(\beta\right) > 0\right).$$

Shukla and Prajapati [12] also derived the following integral

$$\int_0^\infty e^{-st}t^{\beta-1}\frac{d^k}{dz^k}E_{\alpha,\beta}\left(yt^\alpha\right)dt = \frac{k!s^{\alpha-\beta}}{\left(s^\alpha - y\right)^{k+1}}. \tag{2.11}$$

3. Mathematical formulation of the problem

Let a long circular cylinder in which fluid is at rest initially and a constant pressure gradient is imposed along the axis of the cylinder, due to the pressure gradient fluid is set into the motion (constant ρ and μ). Let Z as the direction of the axis of cylinder along which the flow takes place and let r be the radial direction outward from the Z-axis, consider the flow is fully developed and axially symmetric. Here, we assume that there are some depositions of thickness δ on the wall of the cylinder which causes the narrowing the system, which satisfies the following equation of the thickness due to deposition:

$$R = R_0 - \frac{\delta}{2}\left(1 + \cos\frac{\pi z}{z_o}\right),$$

where δ is the deposition thickness, R_0 is the distance from axis of the cylindrical boundary and z is the distance from $z = 0$ to the point of calculation P.

If $z = 0$ then $R = R_0 - \delta$ and if $z = z_0$ then $R = R_0$.

Since velocity u and v are zero, pressure depends on z then we arrive at the conclusion by using equations (1.3), (1.4) and (1.5):

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

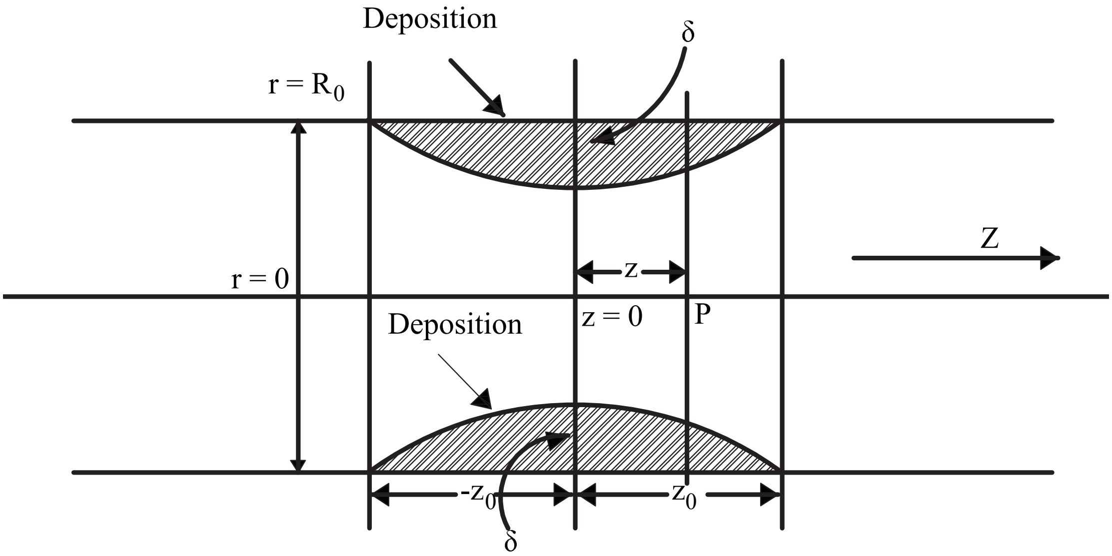


Figure 1: Schematic diagram of narrowing system

Here we consider the velocity component is invariant in the θ and z directions, then above equation reduces to:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)$$

Now, applying the equation of continuity

$$\frac{\partial w}{\partial z} = 0,$$

the Z-momentum equation can be written in a simplified form as:

$$\rho\frac{\partial w}{\partial t} = P + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3.12}$$

where μ is dynamic viscosity and

$$P = -\frac{\partial p}{\partial z}.$$

Initial condition and boundary conditions are considered as:

$$\begin{cases} w(r,0) = 0 \\ w(R,t) = 0 \\ w(0,t) \text{ is finite} \end{cases} \tag{3.13}$$

4. Solution of the problem

The method of integral transform is used to obtain the solution of the problem. Let

$$\bar{w}(\lambda_n, t) = H_0(w(r,t)) = \int_0^R r\, w(r,t)\, J_0(r\lambda_n)\, dr \tag{4.14}$$

where λ_n are the roots of the equation $J_0(R\lambda_n) = 0$. Also, by using the recurrence relation of the Bessel function, we have

$$\int_0^R r\, J_0(r\lambda_n)\, dr = \frac{R}{\lambda_n} J_1((R\lambda_n). \tag{4.15}$$

By taking the zero order finite Hankel transform (2.8) of (3.12) and using (4.14), (4.15) & (3,13) , yields

$$\rho \frac{\partial \bar{w}}{\partial t} = \frac{PRJ_1(R\lambda_n)}{\lambda_n} - \mu\lambda_n^2 \bar{w}. \tag{4.16}$$

Let

$$\tilde{\bar{w}}(\lambda_n, s) = L\{\bar{w}(\lambda_n, t)\} = \int_0^\infty e^{-st} \bar{w}(\lambda_n, t)\, dt. \tag{4.17}$$

Also,

$$\int_0^\infty e^{-st} dt = \frac{1}{s}. \tag{4.18}$$

By taking the Laplace transform (2.6) of (4.16) and using (4.17), (4.18) & (3.13), we get

$$\rho s \tilde{\bar{w}}(s) = \frac{PRJ_1(R\lambda_n)}{\lambda_n s} - \mu\lambda_n^2 \tilde{\bar{w}}.$$

Further simplification gives

$$\tilde{\bar{w}} = \frac{PRJ_1(R\lambda_n)}{(\rho s + \mu\lambda_n^2)\lambda_n s}. \tag{4.19}$$

Now, taking the inverse Laplace transform of this equation, gives

$$\bar{w} = \frac{PRJ_1(R\lambda_n)}{\lambda_n \rho} L^{-1}\left\{ \frac{1}{s\left(s + \frac{\mu}{\rho}\lambda_n^2\right)} \right\}.$$

Now, using Convolution theorem (Debnath [8]), we have

$$\bar{w} = \frac{PRJ_1(R\lambda_n)}{\lambda_n \rho} \int_0^t L^{-1}\left\{ \frac{1}{\left(s + \frac{\mu}{\rho}\lambda_n^2\right)}, u \right\} du,$$

and using (2.11), we get:

$$\bar{w} = \frac{PRJ_1\left(R\lambda_n\right)t}{\lambda_n \rho} E_{1,2}\left(-\lambda_n^2 \frac{\mu}{\rho} t\right). \tag{4.20}$$

We can easily verify that

$$tE_{1,2}\left(mt\right) = \frac{1}{m}\left[e^{mt} - 1\right]$$

and putting this result in (4.20) afterwards taking the inverse finite Hankel transform yields

$$w = -\frac{2}{R^2}\sum_{n=1}^{\infty}\left\{\frac{PRJ_1\left(R\lambda_n\right)}{\lambda_n \rho}\frac{1}{\lambda_n^2 \frac{\mu}{\rho}}\left[e^{-\lambda_n^2 \frac{\mu}{\rho} t} - 1\right]\right\}\frac{J_0\left(r\lambda_n\right)}{J_1^2\left(R\lambda_n\right)}.$$

Further simplification of this result becomes in following form,

$$w\left(r,t\right) = \frac{P}{4\mu}\left(R^2 - r^2\right) - \frac{2P}{\mu R}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n r\right)}{\lambda_n^3 J_1\left(\lambda_n R\right)} e^{\left(-\frac{\mu}{\rho}\lambda_n^2 t\right)}. \tag{4.21}$$

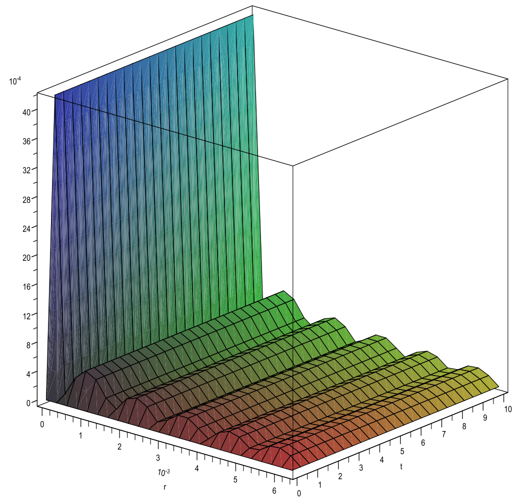
If $t \to \infty$, then $w\left(r,t\right) = \dfrac{P}{4\mu}\left(R^2 - r^2\right)$.

5. Conclusion

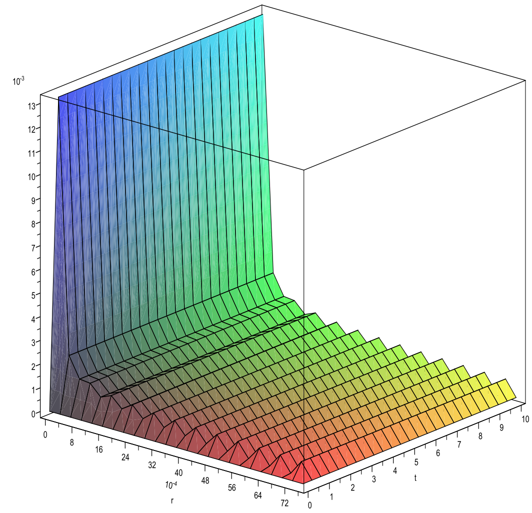
In this paper, we obtained the analytic solution of fluid flow through narrow system in the terms of Bessel function and Mittag-Leffler function by applying Laplace transform and finite Hankel transform techniques. The behavior of the flow has also been shown in the graphs for different values of operational radius R.

By using $P = 101325$ Pa, $\rho = 1000$ Ns/m^2, $\mu = 0.0010020$ kg/m^3 and $R_0 = 0.0127$ m in

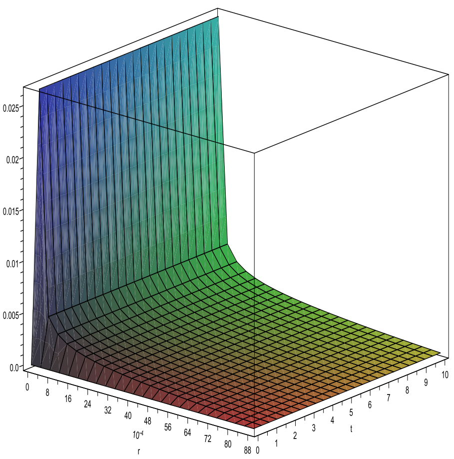
$$w\left(r,t\right) = \frac{P}{4\mu}\left(R^2 - r^2\right) - \frac{2P}{\mu R}\sum_{n=1}^{\infty}\frac{J_0\left(\lambda_n r\right)}{\lambda_n^3 J_1\left(\lambda_n R\right)} e^{\left(-\frac{\mu}{\rho}\lambda_n^2 t\right)}.$$



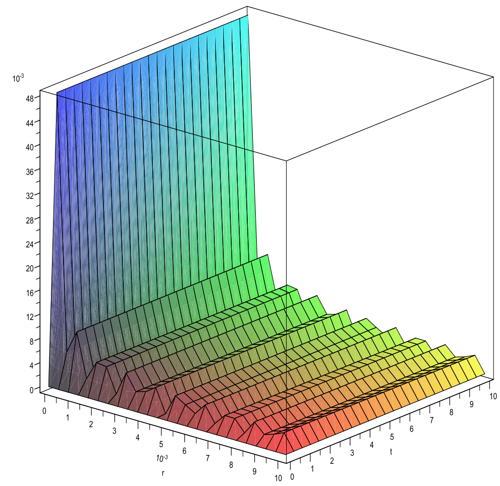
(a) $R = 0.00635$



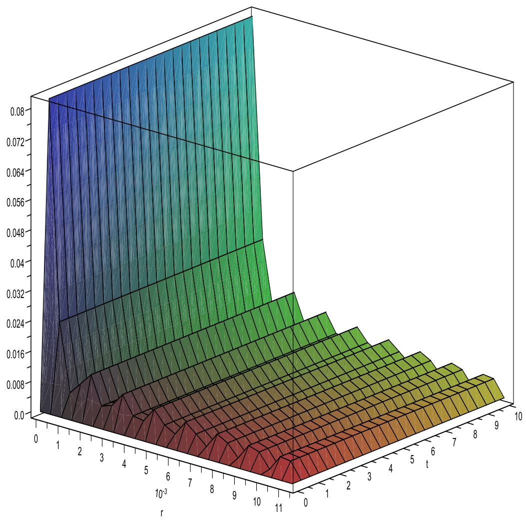
(b) $R = 0.00762$



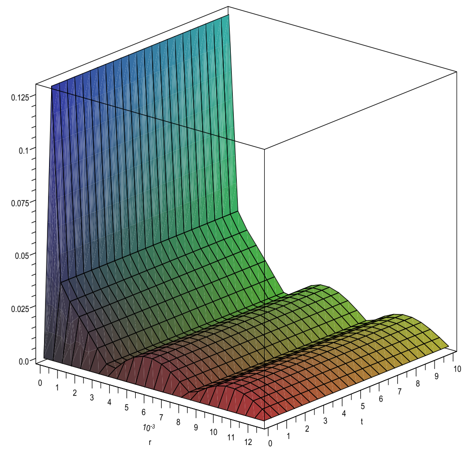
(c) R=0.00889



(d) R=0.01016



(e) R=0.01143



(f) R=0.0127

Figure 2: Velocity profile for different operational radius R.

References

[1] Murlidhar, K. and Gautam Biswas, *Advanced engineering fluid mechanics*, Narosa Publications, 2005.

[2] Verma, V.K., Katiyar, V.K. and Singh, M.P., *Effect of multiple stenosis on blood flow through a tube*, Journal of Natural and physical sciences, 2007.

[3] Verma, V.K., Singh, M.P. and Katiyar, V.K., *Analytical study of blood flow through an artery with mild stenosis*, Acta Ciencia Indica, vol. XXX M, 2 (281), 2004.

[4] Ponalagusamy, R., *Blood flow through an artery with mild stenosis: A two layered model, different shapes of stenosis and slip velocity at the wall*, Journal of Applied Sciences, 7 (7) (2007), 1071.

[5] Chaturani, P. and Kaloni, P.N., *Two-layered poiseuille flow model for blood flow through arteries of small diameter and arterioles*, Biorehology, 13 (1976), 243-250.

[6] Chaturani, P. and Upadhya, V.S., *A two-layered model for blood flow through small diameter tubes*, Biorehology, 16 (1979), 109-118.

[7] Chaturani, P. and Biswas, D., *A theoretical study of blood flow through stenoded arteries with velocity slip at the wall*, Proc. First Internat Symposium on Physiological Fluid Dynamics, IIT Madras, India, 1983, 23-26.

[8] Debnath, L., *Integral Transforms and their Applications*, CRC Press, New York-London-Tokyo, 1995.

[9] Rainville, E.D., *Special Functions*, The Macmillan Company, New York, 1960.

[10] Mittag-Leffler, G.M., *Sur la nouvelle fonction $E\alpha(x)$*, C.R. Acad. Sci. Paris, 137 (1903), 554-558.

[11] Wiman, A., *Über den fundamental Satz in der Theorie der Funktionen $E\alpha(x)$*, Acta Math., 29 (1905), 191-201.

[12] Shukla, A.K. and Prajapati, J.C., *On a generalization of Mittag–Leffler function and its properties*, J. Math. Anal. Appl., 336 (2007), 797-811.

Accepted: 12.09.2009