QUASI-PERMUTATION REPRESENTATIONS OF SOME MINIMAL NON-ABELIAN p-GROUPS

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Abstract. In [1], $c(G)$, $q(G)$ and $p(G)$ are defined for a finite group G. In this paper, we will calculate $c(G)$, $q(G)$ and $p(G)$ for the following minimal non abelian p-groups:

$$G = \langle a, b \mid a^{p^m} = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle,$$

and will show that

$$c(G) = q(G) = p(G) = p\, c(Z(G)) = p^m + p^2.$$

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1. Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [4]. It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

Let G be a non abelian group. G is called a minimal non abelian group, if all its proper subgroups are abelian groups. In [6], all minimal non abelian p-groups are determined as the next Lemma.

Lemma 1.1 *Let G be a minimal non abelian p-group. Then $G = \langle a, b\rangle$, is one of the following groups:*

(1) $G = Q_8$,

(2) $G = \left\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \right\rangle \quad (m > 1)$,

(3) $G = \langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle$.

Furthermore, the last group is not metacyclic and in the case $p = 2$, we have $m \geq n$, $m \geq 2$. Also, $|G| = p^{m+n+1}$, $G' = \langle c\rangle$ and $Z(G) = \langle a^p\rangle \times \langle b^p\rangle \times \langle c\rangle$.

The groups (1) and (2) in the above Lemma are metacyclic and the quasi permutatation representation of such groups has calculated in [2], and [3]. Therefore, to determine the quasi permutation representation of minimal non abelian p-groups it is enough to consider only the group

$$G = \left\langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\right\rangle,$$

and determine $p(G), q(G)$ and $c(G)$. In This paper we consider a special case of this group, that is $n = 1$.

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\right\rangle,$$

2. Comparing the characters of G and $Z(G)$

In this section we state some relations between the characters of G and $Z(G)$, and construct the character table of G. In the next section, these connections will help us to compare the Galois conjugacy classes of irreducible characters of G and $Z(G)$, and conclude that $c(G) \geq p\, c(Z(G))$. Since $Z(G)$ is abelian, so computing $c(Z(G))$ is immediate (see [4]).

Lemma 2.1 *Let*

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\right\rangle,$$

where $m \geq 2$ in the case $p = 2$. Then

a) $|G| = p^{m+2}$,

b) $Z(G) = \langle a^p\rangle \langle c\rangle \cong C_{p^{m-1}} \times C_p$ *and* $|Z(G)| = p^m$,

c) $G' = \langle c\rangle$, *and* $|G'| = p$,

d) $G/G' = \langle aG' \rangle \langle bG' \rangle \cong C_{p^m} \times C_p$ *and* $|G/G'| = p^{m+1}$,

e) $\mathrm{cd}(G) = \{1, p\}$,

f) $|\mathrm{Lin}(G)| = p^{m+1}$ *and* $|\mathrm{Irr}(G|G')| = (p-1)p^{m-1}$, *where by* $\mathrm{Irr}(G|G')$ *we mean the set of non-linear irreducible characters of* G.

g) *The conjugacy classes of* G *are* $A_{i,j}$*'s* $(0 \leq i < p^m,\ 0 \leq j < p$, *and* $p \nmid i$ *or* $j \neq 0)$ *and the central classes , where*

$$A_{i,j} = a^i b^j \langle c \rangle .$$

Proof. a), b), c) and d) are clear by Lemma 1.1.

e) For each $\chi \in \mathrm{Irr}(G)$, we have $\chi(1)^2 \leq |G : Z(G)|$ by [[5], Corollary (2.30)]. So the result follows from (b).

f) It is clear from (d), (e) and the fact that $|G| = \Sigma \chi(1)^2$ where χ runs over $\mathrm{Irr}(G)$.

g) Let x be an arbitrary non central element of G. For each element y in G we have

$$x^y = x[x, y] \in xG' = \{x, xc, xc^2, \ldots, xc^{p-1}\}.$$

Since the order of each conjugacy classes of G divides the order of G, so the result follows. ∎

Lemma 2.2 *Let* ψ *be a linear character of* G. *Denote the restriction of* ψ *to* $Z(G)$, *by* ψ_Z. *Then* ψ_Z *is an irreducible character* χ *of* $Z(G)$ *such that* $\chi(c) = 1$. *Furthermore, for a given* $\chi \in \mathrm{Irr}(Z(G))$ *with* $\chi(c) = 1$, *there are exactly* p^2 *linear character* ψ *of* G *such that* $\psi_Z = \chi$.

Proof. Let $\omega = e^{(2\pi i/p^{m-1})}$, $\mu = e^{(2\pi i/p)}$ and $u = e^{(2\pi i/p^m)}$. Let $\chi_{\upsilon,r}$ be the characters of $Z(G)$, where $0 \leq \upsilon < p^{m-1},\ 0 \leq r < p$ and

$$\chi_{\upsilon,r}(a^p) = \omega^\upsilon,\ \chi_{\upsilon,r}(c) = \mu^r.$$

Also let $\psi_{s,t}$ be the linear characters of G, where $0 \leq s < p^m,\ 0 \leq t < p$ and

$$\psi_{s,t}(a) = u^s,\ \psi_{s,t}(b) = \mu^t.$$

Now, since $\omega = u^p$, so for each $0 \leq \upsilon < p^{m-1}$ we have

$$\psi'_{\upsilon+\alpha p^{m-1},\beta} = \chi_{\upsilon,o} \qquad (1)$$

where $0 \leq \alpha, \beta < p$ and by ψ' we mean ψ_Z. Note that the characters $\chi_{\upsilon,o}$ are exactly the characters of $Z(G)$ such that they have value 1 on c. ∎

The characters $\chi_{\upsilon,r}$ $(0 < r < p)$ of $Z(G)$ have not the value 1 on c. In the next lemma we connect these characters to the non-linear irreducible characters of G.

Lemma 2.3 *The non-linear irreducible characters of G are exactly the characters* $p\overline{\chi}_{\upsilon,r}$, $(0 \leq \upsilon < p^{m-1}, 0 < r < p)$ *defined as*

$$\overline{\chi}_{\upsilon,r}(x) = \begin{cases} \chi_{\upsilon,r}(x) & \text{if } x \in Z(G) \\ 0 & \text{if } x \notin Z(G) \end{cases} .$$

Proof. By [[5], Lemma (2.27) part (c) and Corollary (2.30)], every non-linear irreducible character ψ of G vanishes on $G - Z(G)$ and $\psi_Z = p\chi$ for some irreducible character χ of $Z(G)$.

Now, to complete the proof, it is enough to show that, for $0 \leq \upsilon < p^{m-1}$, $p\overline{\chi}_{\upsilon,o}$, can not be an irreducible non-linear character of G. To prove this, let us compute the inner product of $p\overline{\chi}_{\upsilon,o}$ and $\psi_{\upsilon,o}$:

$$\begin{aligned} \langle p\overline{\chi}_{\upsilon,o}, \psi_{\upsilon,o} \rangle &= \frac{1}{|G|} \sum_{g \in G} p\overline{\chi}_{\upsilon,o}(g) \psi_{\upsilon,o}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in Z(G)} p\chi_{\upsilon,o}(g) \psi'_{\upsilon,o}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in Z(G)} p\chi_{\upsilon,o}(g) \chi_{\upsilon,o}(g^{-1}) \\ &= p \, \frac{|Z(G)|}{|G|} \frac{1}{|Z(G)|} \sum_{g \in Z(G)} \chi_{\upsilon,o}(g) \chi_{\upsilon,o}(g^{-1}) \\ &= p \, \frac{|Z(G)|}{|G|} \, \langle \chi_{\upsilon,o}, \chi_{\upsilon,o} \rangle \\ &= p \, \frac{|Z(G)|}{|G|} = \frac{1}{p} \neq 0. \end{aligned}$$

Note that $p\overline{\chi}_{\upsilon,o}$ vanishes on $G - Z(G)$ and $\psi' = \psi_Z$, so the second equality holds. For the third equality, put $\alpha = \beta = 0$ in the relation (1). Hence $p\overline{\chi}_{\upsilon,o}$, is not even a character of G by [[5], Corollary (2.17)]. ■

Theorem 2.4 *Let*

$$G = \left\langle a, b \mid a^{p^m} = b^p = c^p = 1, \; [a,b] = c, \; [a,c] = [b,c] = 1 \right\rangle .$$

Then the character table of G has the form

	$Z(G)$	$a^i b^j$
$\psi_{\upsilon+\alpha p^{m-1},\beta}$	$\chi_{\upsilon,0}$	$u^{i\upsilon}\mu^{i\alpha+j\beta}$
$p\overline{\chi}_{\upsilon,r}$	$p\chi_{\upsilon,r}$	0

where $(0 \le i < p^m,\ 0 \le j < p,\ p \nmid i$ *or* $j \neq 0)$ *and* $(0 \le v < p^{m-1},\ 0 < r < p,$ $0 \le \alpha, \beta < p)$.

Proof. This table follows by Lemmas 2.1, 2.2 and 2.3. Note that, for $0 \le s < p^m$, $0 \le t < p$, we have

$$\psi_{s,t}(a^i b^j) = u^{si}\mu^{tj}.$$

Also, note that

$$u^{p^{m-1}} = \mu.$$

■

3. Computing $c(G)$, $q(G)$ and $p(G)$

First, we state some notations and algorithms from [1]. Let G be a finite group. Let $\mathcal{C}_i$ for $0 \le i \le r$ be the Galois conjugacy classes of irreducible complex characters of the group G over the rational field $\mathbb{Q}$, for $0 \le i \le r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$, and $K_i = \ker \psi_i$. Clearly, $K_i = \ker \Psi_i$. For $I \subseteq \{0, 1, ..., r\}$, put

$$K_I = \bigcap_{i \in I} K_i.$$

Also, if $I \subseteq \{1, ..., r\}$, then we will use the notation $m(\chi)$, where $\chi = \sum_{i \in I} \Psi_i$ and $m(\chi) = -\min \left\{ \sum_{i \in I} \Psi_i(g) : g \in G \right\}$. Moreover let $H_G = \bigcap_{g \in G} H^g$ be the core of $H \le G$.

Theorem 3.1 *Let G be a finite group. Then in the above notation*

$$c(G) = \min\Big\{\chi(1) + m(\chi) : \chi = \sum_{i \in I} \Psi_i, K_I = 1,\ I \subseteq \{1, ..., r\}, K_J \neq 1, \text{if } J \subset I\Big\}$$

$$p(G) = \min\Big\{ \sum_{i=1}^{n} |G : H_i| : H_i \le G \text{ for } i = 1, 2, ..., n \text{ and } \bigcap_{i=1}^{n} (H_i)_G = 1\Big\}.$$

Proof. See [[1], Theorems (2.2) and the Proof of (3.6)]. ■

Lemma 3.2 *Let G be as in Lemma 2.1 and let we have the same notations as in Lemma 2.2 for irreducible characters of $Z(G)$. Then the Galois conjugacy classes of* $\mathrm{Irr}(Z(G))$ *are* $\mathcal{C}_0 = \{\chi_{0,0}\}$ *and*

$$\mathcal{C}_{p^j} = \{\chi_{ip^j,0} : 1 \le i \le p^{m-1-j} - 1, (i, p^{m-1}) = 1\},$$

where $0 \le j \le m-2$, *and the Galois conjugacy classes of the characters* $\chi_{v,r}$, $r \neq 0$.

Proof. First, note that no $\chi_{v,r}$ can be a Galois conjugate to some $\chi_{v',0}$ when $r \neq 0$, because of their different values on c. So, we consider only the characters $\chi_{v,0}$. Since $Z(G) = \langle a^2 \rangle \langle c \rangle$ and $\chi_{v,0}(\langle c \rangle) = 1$, so there is a one to one corresponding between the set of characters $\chi_{v,0}$ of $Z(G)$ and the set of the characters of the cyclic group $\langle a^2 \rangle \cong C_{p^{m-1}}$ via the map $\chi_{v,0} \mapsto \chi_v$. Clearly, this map preserves Galois conjugates. Now let $\Gamma(\chi)$, denote the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$. Then, for $0 \leq j \leq m-2$, we have

$$\Gamma(\chi_{p^j,0}) = \Gamma(\chi_{p^j}) = \mathrm{Gal}(\mathbb{Q}(\chi^{p^j})/\mathbb{Q}) \text{ and}$$

$$\Gamma(\chi_{p^j}) = \{\sigma_i : \sigma_i \text{ is an } \mathbb{Q}-\text{automorphism of } \mathbb{Q}(\omega^{p^j}) \text{ and } \sigma_i(\omega^{p^j}) = \omega^{ip^j}\},$$

where $1 \leq i \leq p^{m-1-j}-1, (i, p^{m-1}) = 1$. This shows that $\mathcal{C}_{p^j}$, is the Galois conjugacy class of the character $\chi_{p^j,0}$. The order of the class $\mathcal{C}_{p^j}$ is equal to $(p-1)p^{m-2-j}$. These classes are all different and counting their elements shows that they are all Galois conjugacy classes of $Z(G)$. ∎

Lemma 3.3 *Let G be as in Lemma 2.1 and let we have the same notations as in Lemma 2.2 and 2.3 for irreducible characters of G. Then the Galois conjugacy classes of $\mathrm{Irr}(G)$ are*

(1) *The Galois conjugacy classes of the characters $\psi_{\alpha p^{m-1},\beta}$ where $0 \leq \alpha, \beta < p$,*

(2) *The Galois conjugacy classes*

$$\mathcal{C}^{(\beta)}_{p^j} = \{\psi_{ip^j+\alpha p^{m-1},\overline{i\beta}} : 1 \leq i \leq p^{m-1-j}-1, (i,p) = 1, 0 \leq \alpha < p\},$$

where $0 \leq j \leq m-2$, $0 \leq \beta < p$ and by $\overline{i\beta}$ we mean $i\beta$ module p,

(3) *The Galois conjugacy classes of the characters $p\overline{\chi}_{v,r}, r \neq 0$.*

Proof. Each character in (1) is 1 on $Z(G)$, so it can not be a Galois conjugate to any character in (2) or (3). The characters in (3) have degree p, so these characters can not be Galois conjugate to (linear) characters in (1) or (2). Thus, we consider only the characters in (2). We show that, for $0 \leq j \leq m-2$, $\mathcal{C}^{(\beta)}_{p^j}$ is the Galois conjugacy class of the character $\psi_{p^j,\beta}$. Let $\psi_{ip^j+\alpha p^{m-1},\overline{i\beta}} \in \mathcal{C}^{(\beta)}_{p^j}$. Let $\tau \in \Gamma(\psi_{p^j,\beta}) = \mathrm{Gal}(\mathbb{Q}(u^{p^j}, \mu^\beta)/\mathbb{Q})$ and $(u^{p^j})^\tau = (u^{p^j})^{i+\alpha p^{m-1-j}}$. Then $(u^{p^j})^\tau = (u^{p^j})^i \mu^\alpha$ and $(\mu^\beta)^\tau = \mu^{i\beta}$. Therefore, $\psi^\tau_{p^j,\beta} = \psi_{ip^j+\alpha p^{m-1},\overline{i\beta}}$. On the other hand, $|\Gamma(\psi_{p^j,\beta})| = \varphi(p^{m-j}) = (p-1)p^{m-j-1} = |\mathcal{C}^{(\beta)}_{p^j}|$ where φ is the Euler function. Therefore, $\mathcal{C}^{(\beta)}_{p^j}$ is the Galois conjugacy class of the character $\psi_{p^j,\beta}$. Counting the elements of these classes shows that they are, in addition to classes of (1) and (3), all Galois conjugacy classes of G. ∎

Lemma 3.4 *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then $\Sigma_{\alpha \in \Gamma(\chi)} \chi^\alpha$ is a rational valued character of G.*

Proof. By [[1], Corollary 3.7]. ∎

Now, as the notations introduced before Lemma 3.1, let $\Psi_{p^j} = \sum \mathcal{C}_{p^j}$, $\Psi_{p^j}^{(\beta)} = \sum \mathcal{C}_{p^j}^{(\beta)}$ where $0 \leq j < m-2$. Let Υ_i's be the sum's of Galois conjugacy classes of characters (1) in the Lemma 3.3. Also, let Φ_i's be the sums of Galois conjugacy classes of characters $\chi_{v,r}, r \neq 0$ and Φ_i''s be the sums of Galois conjugacy classes of characters $p\overline{\chi}_{v,r}$. Note that these sums are rational valued characters by Lemma 3.4 and $\Psi_{p^j}^{(\beta)} = \sum \mathcal{C}_{p^j}^{(\beta)} = p \sum \mathcal{C}_{p^j} = p\Psi_{p^j}$ on $Z(G)$. Also $\Phi_i' = p\,\Phi_i$ on $Z(G)$ and $\Phi_i' = 0$ on $G - Z(G)$.

Theorem 3.5 *Let G be as in Lemma* 2.1. *Then, in the algorithm given in Theorem* 3.1, *the classes Υ_i are not used for computing $c(G)$. Therefore,*

$$c(G) = q(G) = p(G) = p\,c(Z(G)) = p^m + p^2.$$

Proof. Let $\mathbb{S}$ be a subset of the set of class sums Υ_i that has used in computing $c(G)$ with a set $\mathbb{T}$ of other class sums. Since $Z(G) \subseteq \bigcap_{S \in \mathbb{S}} \ker S$, so by the algorithm of $c(G)$, given in Theorem 3.1, $\mathbb{T} \neq \varnothing$. Now there is an element T_i of $\mathbb{T}$ such that T_i is vanishes on $G - Z(G)$, because otherwise, $c \in \bigcap_{T \in \mathbb{T}} \ker T$, that is a contradiction. Therefore the kernels of the elements of $\mathbb{T}$ have no intersection in $G - Z(G)$, and clearly no non-trivial intersection in $Z(G)$. Thus

$$\bigcap_{T \in \mathbb{T}} \ker T = 1.$$

This is a contradiction to the choice of the sets $\mathbb{S}$ and $\mathbb{T}$. Therefore, $\mathbb{S} = \varnothing$.

Now, by the algorithm of $c(G)$ given in Theorem 3.1 and the argument after Lemma 3.4, we conclude that $p\,c(Z(G)) \leq c(G)$. In other hand, let $H_1 = \langle a \rangle$ and $H_2 = \langle b, c \rangle$ then, by Theorem 3.1,

$$p(G) \leq |G : H_1| + |G : H_2| = p^2 + p^m.$$

Since $p\,c(Z(G)) = p^2 + p^m$ by [[4], Theorem A], so

$$c(G) = q(G) = p(G) = p\,c(Z(G)) = p^m + p^2,$$

and the proof is complete. ∎

References

[1] Behravesh, H., *Quasi-permutation representations of p-groups of class* 2, J. London Math. Soc. (2) 55 (1997) 251-260.

[2] Behravesh, H., *Quasi-permutation representations of metacyclic 2-groups*, J. Sci. I. R. Iran. 9(3) (1998) 258-264.

[3] Behravesh, H., *Quasi-permutation representations of metacyclic p-groups with non-cyclic center*, Southeast Asian Bull. Math. Springer-Verlag 24 (2000) 345-353.

[4] Burns, J.M., Goldsmith, B., Hartley, B., Sandling, R., *On quasi-permutation representations of finite groups* , Glasgow Math. J. 36 (1994) 301-308.

[5] Isaacs, I.M., *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[6] Redei, L., *Endliche p-Groupen*, Budapest: Akademiai Kiado, 1989 (III, Satz 6.1, p. 291.)

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