

ON INJECTIVITY OF PROJECTION AND SEPARATED PROJECTION ALGEBRAS

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Abstract. Projection spaces (algebras) were first introduced by Ehrig et. al. as an algebraic version of ultrametric spaces, and then studied by Giuli, Ebrahimi, Mahmoudi. Computer scientists use projection algebras for algebraic specification of process algebras. A kind of *injectivity* of separated projection algebras have been studied by Giuli. In this paper, we extend this notion to all projection algebras, and introduce some other kind of injectivity, so called m and p -injectivity, and show, among other things, that injectivity, s -injectivity, and m -injectivity coincide, and so we get some more Baer criteria for injectivity.

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1. Introduction and preliminaries

The notion of a projection space (algebra) was first introduced by Ehrig et. al. as an algebraic version of an ultrametric space ([10]). Computer Scientists use this notion for a formal description of parallel concurrent systems. One of the main problem in this scope is the specification of infinite objects (processes) which can not be denoted by finite terms. So, they use projection algebras as a convenient means for algebraic specification of process algebras (see [10], [11] and their references). Projection algebras have also, naturally, been studied by mathematicians, for example in [7], [8], [9], [12], [14].

In this paper we consider three types of closure operators on projection algebras to get the classes of dense and closed monomorphisms naturally arising from them. Then we study injectivity with respect to these classes of monomorphisms in the categories of projection and separated projection algebras arising from these closure operators (see also, [8], [12], [14]).

We now officially recall the category **PRO** of projection algebras.

A *projection space* (considering it as a kind of universal algebra we prefer to call it a *projection algebra*) is in fact a (right) M -set (or M -act) for the monoid $M = \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with the binary operation $m.n = \min\{m, n\}$, where \mathbb{N} is the set of natural numbers and $n < \infty, \forall n \in \mathbb{N}$. In other words, it is a set A together with a family $(\lambda_n)_{n \in \mathbb{N}^\infty}$ of unary operations $\lambda_n : A \rightarrow A$ (called projections) such that

$$\lambda_m \circ \lambda_n = \lambda_{n.m} \text{ and } \lambda_\infty = id_A$$

for every $m, n \in \mathbb{N}^\infty$. We denote $\lambda_n(a)$ by na , for $n \in \mathbb{N}^\infty, a \in A$.

A *projection morphism* between projection algebras is also called an *equivariant map*. In fact, a projection morphism between projection algebras $(A, (\lambda_n)_{n \in \mathbb{N}^\infty})$ and $(B, (\eta_n)_{n \in \mathbb{N}^\infty})$ is a function $f : A \rightarrow B$ satisfying $f \circ \lambda_n = \eta_n \circ f$, for every $n \in \mathbb{N}^\infty$, that is, $f(na) = nf(a)$, for every $n \in \mathbb{N}^\infty$ and $a \in A$.

Thus, the category **PRO** of projection algebras is a special kind of the category **MSet** of M -sets (or **MAct** of acts over M , as in [13]), taking the monoid $M = \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $m.n = \min\{m, n\}$.

The category **PRO** has free objects. In fact, for each set X , $F(X) = \mathbb{N}^\infty \times X$ with actions given by $s(n, x) = (sn, x)$, for $s, n \in \mathbb{N}^\infty$ and $x \in X$, is the free projection algebra generated by X . Also, for each set X , the cofree projection algebra generated by X is the set $H(X) = X^{\mathbb{N}^\infty}$, of all functions from \mathbb{N}^∞ to X , with actions given by $(sf)(n) = f(ns)$, for $f \in X^{\mathbb{N}^\infty}$ and $s, n \in \mathbb{N}^\infty$. In other words, the underlying set functor $U : \mathbf{PRO} \rightarrow \mathbf{Set}$ has a left adjoint F and a right adjoint H (see [6]).

The following proposition is in fact a consequence of the existence of cofree and free projection algebras.

Lemma 1.1. *In the category **PRO** we have:*

- (1) *Epimorphisms are exactly surjective projection morphisms.*
- (2) *Monomorphism are exactly one-one projection morphisms.*
- (3) *Isomorphisms are exactly surjective and injective projection morphisms.*

Proof. (1) Suppose $f : A \rightarrow B$ is an epimorphism in **PRO**. Consider the Rees factor projection algebra B/Imf , that is $[b] = Imf$ for $b \in Imf$ and $[b] = \{b\}$ for $b \in B - Imf$. Define $g : B \rightarrow B/Imf$ by $g(b) = Imf$, for all $b \in B$. Also consider the natural epimorphism $\gamma : B \rightarrow B/Imf$ by $\gamma(b) = [b]$, for $b \in B$. Then we have $\gamma f = gf$, and hence $\gamma = g$, since f is an epimorphism. Thus for all $b \in B$, $[b] = Imf$, that is, $b \in Imf$. Therefore $Imf = B$.

(2) If $f : A \rightarrow B$ is a non injective monomorphism in **PRO**, then there exist a, b in A such that $f(a) = f(b)$ and $a \neq b$. Define $g, h : \mathbb{N}^\infty \rightarrow A$ by $g(n) = na, h(n) = nb$ for $n \in \mathbb{N}^\infty$. Then we have $fg = fh$ while $g \neq h$ because $g(\infty) = a \neq b = h(\infty)$. This contradicts the fact that f is a monomorphism.

(3) is a corollary of (1) and (2). ■

Thus, we can consider monomorphisms $A \rightarrow B$ in **PRO** as inclusions and denote it by $A \leq B$. A projection algebra B containing (a copy of) a projection

algebra A as a subalgebra is called an *extension* of A . The algebra A is said to be a *retract* of B if there exists a homomorphism $f : B \rightarrow A$ such that $f|_A = id_A$, in which case f is said to be a *retraction*. A is called *absolute retract* if it is a retract of each of its extensions.

Note that each non-empty projection algebra A has a *zero (fixed)* element, that is an element a_0 with $sa_0 = a_0$, for each $s \in \mathbb{N}$. In fact, for each $a \in A$, $1a$ is a zero element of A .

Finally, we mention the following from general results of [2] or [5], which will be needed in this paper.

Proposition 1.2. *The category PRO has enough injectives, and consequently injectives and absolute retracts coincide in PRO.*

Now, we use projection algebras to introduce projection specifications which are useful for computer scientists. But we will not use this version of projection algebras in this paper.

Definition 1.3. A *signature* is a pair $SIG = (S, OP)$, where S is a set of sorts (set symbols) and OP is a set of constants and operation symbols.

An algebra of a signature $SIG = (S, OP)$ or a *SIG-algebra* is a pair $A = (S_A, OP_A)$ where S_A is a family $(A_s)_{s \in S}$ of sets called base sets or domain of A , and OP_A is a family $(N_A)_{N \in OP}$ of elements of A_s for all constant symbols $N : \rightarrow s$ and $s \in S$, called constants of A , or functions $N_A : A_{s_1} \times A_{s_2} \times \dots \times A_{s_n} \rightarrow A_s$ for all operation symbols $N \in OP_{s_1 \dots s_n, s}$ and $s_1 \dots s_n \in S^* \setminus \{\lambda\}$, $s \in S$, called operations of A .

A *specification* $SPEC = (S, OP, E)$ consists of a signature $SIG = (S, OP)$ and a set E of equations with respect to SIG .

An algebra of a specification $SPEC = (S, OP, E)$ or a *SPEC-algebra* is an algebra A of the signature SIG which satisfies all the equations in E .

Example 1.4. This is the specification *nat1* of the natural numbers starting with 1.

$$\begin{aligned}
 \text{nat1} &= \\
 \text{sorts} &: \text{nat1} \\
 \text{opns} &: 1 : \rightarrow \text{nat1} \\
 &\quad \text{succ} : \text{nat1} \rightarrow \text{nat1} \\
 &\quad \text{min} : \text{nat1} \text{ nat1} \rightarrow \text{nat1} \\
 \text{eqns} &: \text{for all } m, n \text{ in } \text{nat1} : \\
 &\quad \text{min}(n, 1) = 1 \\
 &\quad \text{min}(1, n) = 1 \\
 &\quad \text{min}(\text{succ}(n), \text{succ}(m)) = \text{succ}(\text{min}(n, m))
 \end{aligned}$$

Definition 1.5. A *projection specification* $SPEC = (S, OP, E)$ is an algebraic specification with

- (i) $\text{nat1} \subseteq SPEC$,
- (ii) For all $s \in S$ there is an operation symbol $ps : \text{nat1 } s \rightarrow s \in OP$, and $p\text{nat1}(n, k) = k$, for all $n, k \in \text{nat1}$,

- (iii) $nat1 \notin rang(OP - (\{pnat1 : nat1 \ nat1 \rightarrow nat1\} \cup OP((nat1))))$ That is there is no operation symbols $N : s1...sn \rightarrow nat1$ in OP except the operation $pnat1$ and the operation symbols from $nat1$,
- (iv) if $t1 = t2 \in E - E(nat1)$ then $sort(t1) \neq nat1$, that is there is no additional equations between $nat1$ -terms.

A *projection-Spec-algebra* is an algebra of the specification $SPEC$ with the additional properties

- (i) For all $s \in S$, (A_s, ps_A) is a projection space,
- (ii) the equations N_A are projection compatible, that is $ps_A(k, N_A(a1, \dots, an)) = ps_A(k, N_A(ps_{1_A}(k, a1), \dots, ps_{n_A}(k, an)))$, for all $N : s1...sn \rightarrow s$, for all $k \geq 1$, for all $a1 \in A_{s1}, \dots, an \in A_{sn}$,
- (iii) $A_{nat1} \simeq N_1$.

2. Some closure operators on PRO

In this section we give three kinds of closure operators on the category **PRO**, two of which have already been introduced in [12], [3]. We use these closure operators to define separated projection algebras in the next section. Although some of the results are consequences of the general results of [4], we give direct proofs.

Definition 2.6. For a projection algebra B and each subalgebra A of B define:

- (1) The m -closure of A in B , for $m \in \mathbb{N}$, by $C_m(A \leq B) := \{b \in B : kb \in A, \forall k \leq m\}$.
- (2) the s -closure of A in B by $C_s(A \leq B) := \{b \in B : nb \in A, \forall n \in \mathbb{N}\}$.
- (3) the p -closure of A in B by $C_p(A \leq B) := \{b \in B : \exists a \in A, na = nb, \forall n \in \mathbb{N}\}$.

If there is no confusion, we write $C(A)$ instead of $C(A \leq B)$, for $C \in \{C_m, C_s, C_p\}$.

Note 2.7. It is easily seen that for $m \in \mathbb{N}$ and $A \leq B$ in **PRO**,

$$C_m(A) = \{b \in B : mb \in A\} = \{b \in B : mb \in mA\}$$

Lemma 2.8. Each $C \in \{C_m, C_s, C_p\}$ is an idempotent, hereditary, weakly hereditary, additive, and grounded closure operator on the category **PRO**, in the sense of [4]. That is, for each projection algebra B , we have the following:

- (cl 1) $A \leq C(A)$, for $A \leq B$,
- (cl 2) $A \leq A' \leq B \Rightarrow C(A \leq B) \leq C(A' \leq B)$,

- (cl 3) $f(C(A \leq B)) \leq C(f(A) \leq B')$, for every projection map $f : B \rightarrow B'$,
- (idem) $C(C(A)) = C(A)$, for $A \leq B$,
- (hered) $C(A' \leq A) = C(A' \leq B) \cap A$, for $A' \leq A \leq B$,
- (w-hered) $C(A \leq C(A)) = C(A)$, for $A \leq B$,
- (add) $C(A \cup A') = C(A) \cup C(A')$, for $A, A' \leq B$,
- (ground) $C(\emptyset) = \emptyset$.

Proof. The only part which may be a little tricky is additivity of C_s . In fact, this follows from the fact that for $b \in B$, if $nb \in A \cup A'$, for all $n \in \mathbb{N}$, then two cases may occur: $nb \in A$, for all $n \in \mathbb{N}$, or $nb \in A'$, for all $n \in \mathbb{N}$. This is because, if for some $m \in \mathbb{N}$, $mb \notin A$ (and so $mb \in A'$), then for all $n \in \mathbb{N}$, $nb \in A'$. This is because, if $n \leq m$ then $nb = (nm)b = n(mb) \in A'$, and if $n > m$ then again $nb \in A'$ (because if $nb \in A$ then $mb \in A$ which is contradiction). ■

For closure operators C and D on **PRO**, define $C \leq D$ if for each $A \leq B$ in **PRO**, $C(A) \leq D(A)$. Then, we have the following strict inequalities.

Lemma 2.9. $C_p < C_s < \dots < C_m < \dots < C_2 < C_1$

Proof. It is enough to show that all the above inclusions are proper. To see this, take $A = \downarrow k$ and $B = \mathbb{N}^\infty$, then $C_k(A) = \mathbb{N}$ but $C_{k+1}(A) = \downarrow k$. Also, taking $A = \mathbb{N}$ and $B = \mathbb{N}^\infty$, we have $C_s(A) = \mathbb{N}^\infty$ but $C_p(A) = \mathbb{N}$. ■

Definition 2.10. For a closure operator $C \in \{C_m, C_s, C_p\}$ on **PRO**, a subalgebra A of a projection algebra B is called

- (1) C -closed (respectively, m -closed, s -closed, p -closed) if $C(A) = A$,
- (2) C -dense (respectively, m -dense, s -dense, p -dense) if $C(A) = B$.

A projection map $f : A \rightarrow B$ is called C -dense (C -closed) if $f(A)$ is a C -dense (C -closed) subalgebra of B .

As a corollary of Lemma 2.9, we have

Lemma 2.11. For a projection algebra B and a subalgebra A of B ,

- (1) If A is m -closed then it is k -closed, for all $k \leq m$, and A is s -closed. Also, the latter implies A is p -closed.
- (2) If A is p -dense then it is s -dense. Also, the latter implies A is m -dense. And, if A is m -dense then it is k -dense, for all $k \geq m$.

Notice that, the above lemma holds for morphisms, too.

Theorem 2.12. For each $C \in \{C_m, C_s, C_p\}$, every projection map has (C -dense morphism, C -closed monomorphism) factorization.

Proof. Let $A \rightarrow B$ be a projection map. Take $D = C(f(A) \leq B)$, $g = f : A \rightarrow D$ and $h = \iota : D \rightarrow B$. Then $f = hg$ is a (C -dense, C -closed) factorization of f . ■

3. C -separated projection algebras

Closure operators have been used to generalize the well-known fact that a topological space X is Hausdorff if and only if the diagonal Δ_X is C -closed in $X \times X$. Here, we study the full subcategories

$$\Delta(C) = \{A : \Delta_A \text{ is } C\text{-closed in } A \times A\}$$

for $C \in \{C_m, C_s, C_p\}$.

Definition 3.13. For $C \in \{C_m, C_s, C_p\}$, the subcategories $\Delta(C)$ of **PRO** are denoted by **PRO_m**, **PRO_s**, **PRO_p**, respectively. The elements of these categories are called m (respectively, s , or p)-separated projection algebras.

Example 3.14. Clearly, every projection algebra A with identity actions $\lambda_n = id_A$ is a m -separated; \mathbb{N}^∞ as a projection algebra is s -separated; $A = \{0, 1\}$ in which 0, 1 are zero elements is m -separated as well as s -separated.

Using the definitions of closure operators C_m and C_s , we easily get the following.

Lemma 3.15. *A projection algebra A is*

- (1) *m -separated if and only if $mx = my$ implies that $x = y$, for $x, y \in A$,*
- (2) *m -separated if and only if $ma = a$, for all $a \in A$; if and only if $mA = A$,*
- (3) *s -separated if and only if $nx = ny$, for all $n \in \mathbb{N}$, implies $x = y$.*

Proof. We just prove (2), the rest are straightforward. Let A be m -separated and $ma = a'$, for some $a \in A$. Then $mma = ma'$ and hence $ma = ma'$. So $a = a'$, by (1), since A is m -separated. Conversely, let $ma = a$, for all $a \in A$. If $ma = ma'$ then $a = ma = ma' = a'$. Also, it is clear that the second condition is equivalent to $mA = A$. ■

Now, we easily get the following strict inclusions.

Lemma 3.16. *We have the following strict inclusions*

$$\mathbf{PRO}_1 \subset \mathbf{PRO}_2 \subset \cdots \subset \mathbf{PRO}_s$$

Proof. The inclusions are clearly true. To show that they are strict, take $A = \{a, b\}$ with actions given by $na = a$, for $n \in \mathbb{N}^\infty$, and $kb = a$, for $k \leq m$, $kb = b$, for $k \geq m + 1$. Then $A \in \mathbf{PRO}_{m+1}$ but $A \notin \mathbf{PRO}_m$. Also, \mathbb{N}^∞ is s -separated but it is not m -separated, for $m \in \mathbb{N}$. ■

We also have the following equality.

Lemma 3.17. $\mathbf{PRO}_s = \mathbf{PRO}_p$.

Proof. By Lemma 2.11, for a projection algebra A , if Δ is s -closed in $A \times A$ then it is p -closed in $A \times A$. So, $\mathbf{PRO}_s \subseteq \mathbf{PRO}_p$. Conversely, let Δ be p -closed in $A \times A$ and $nx = ny$, for all $n \in \mathbb{N}$, and some $x, y \in A$. Then taking $z = x$, we have $n(z, z) = (nx, ny)$, for all $n \in \mathbb{N}$. Hence, $(x, y) \in C_p(\Delta) = \Delta$, that is $x = y$. ■

The following results will be used in the study of injectivity in the next section.

Lemma 3.18.

- (1) In \mathbf{PRO}_m , all morphisms are m -closed and hence s -closed and p -closed. But, here the only m -dense (s -dense, or p -dense) monomorphisms are isomorphisms.
- (2) In \mathbf{PRO}_s , all morphisms are p -closed, but the only p -dense monomorphisms are isomorphisms.

Proof. (1) Let $A \leq B \in \mathbf{PRO}_m$. For $b \in C_m(A)$, $mb \in A$ and so, $mb = mmb \in mA$, but by Lemma 3.15(2), $mA = A$. Thus, $b \in A$ and $C_m(A) = A$. Since $C_m \geq C_s \geq C_p$, $A = C_s(A) = C_p(A)$. Further, if A is m -dense in B then $C_m(A) = B$ and so $A = B$.

(2) Let $A \leq B \in \mathbf{PRO}_s$. For $b \in C_p(A)$, there exists $a \in A$ such that $nb = na$, for $n \in \mathbb{N}$. Then $b = a \in A$, since A is s -separated. So $C_p(A) = A$. If A is s -dense in B then $B = C_p(A) = A$. ■

Lemma 3.19.

- (1) Each projection algebra A has a proper p -dense, and hence s -dense, extension.
- (2) Each s -separated projection algebra A has a proper m -dense extension in \mathbf{PRO}_s .

Proof. (1) Take the extension $B = A \cup \{*\}$, $* \notin A$, of A with actions $n* = a_0$, for a zero element a_0 in A and $n \in \mathbb{N}$, also $\infty* = *$. Then B is a proper p -dense (s -dense) extension of A .

(2) Take the set B as defined in (a) with actions $k* = a_0$, for $k \leq m$, and $n* = *$, for $n \geq m + 1$. Then B is a proper m -dense extension of A in \mathbf{PRO}_s . ■

Theorem 3.20. The categories \mathbf{PRO}_m and $\mathbf{PRO}_s = \mathbf{PRO}_p$ are reflective subcategories of \mathbf{PRO} .

Proof. Define the congruence relations \sim_m, \sim_s on a projection algebra A by

$$a \sim_m b \Leftrightarrow ma = mb; \quad a \sim_s b \Leftrightarrow na = nb, \quad \forall n \in \mathbb{N}$$

Then the natural quotient maps $\gamma_m : A \rightarrow A / \sim_m$, and $\gamma_s : A \rightarrow A / \sim_s$ which take $a \in A$ to $[a]$ are reflection arrows from \mathbf{PRO} to \mathbf{PRO}_m and \mathbf{PRO}_s , respectively.

More precisely, if $f : A \rightarrow B$ is a projection map, where B is an m -separated projection algebra, then (by Decomposition Theorem of maps) there exists a unique projection map $\bar{f} : A/\sim_m \rightarrow B$, $\bar{f}([a]) = f(a)$, with the property that $\bar{f}\gamma_m = f$. Similarly, γ_s is a reflection arrow. ■

The following may also be used to study projectivity, which we will not be studying in this paper.

Theorem 3.21.

- (1) In \mathbf{PRO}_m , \mathbf{PRO}_s , the monomorphisms are exactly one-one projection maps.
- (2) In \mathbf{PRO}_m , the epimorphisms, onto projection maps, and m -dense (s -dense, p -dense) projection maps are the same.
- (3) In \mathbf{PRO}_s , the epimorphisms are exactly s -dense morphisms and the onto projection maps are exactly p -dense morphisms.

Proof. (1) Follows from Theorem 3.20 and the fact that in \mathbf{PRO} the monomorphisms are exactly one-one projection maps. More precisely, if $f : A \rightarrow B$ is a monomorphism in \mathbf{PRO}_m then it is a monomorphism in \mathbf{PRO} because if $g, h : C \rightarrow A$ are projection maps with $fg = fh$ then, by Lemma 3.20, C/\sim_m is m -separated and there are projection maps $\bar{g}, \bar{h} : C/\sim_m \rightarrow A$ with $\bar{g}\gamma_m = g$ and $\bar{h}\gamma_m = h$. Now, $f\bar{g}\gamma_m = f\bar{h}\gamma_m$, and hence $\bar{g}\gamma_m = \bar{h}\gamma_m$, since f is a monomorphism in \mathbf{PRO}_m . Thus, $g = h$, and f is a monomorphism in \mathbf{PRO} and hence one-one. A similar argument is true for \mathbf{PRO}_s .

(2) To show that epimorphisms in \mathbf{PRO}_m are onto, apply the same proof as Lemma 1.1 for epimorphisms in \mathbf{PRO} . Further, m -dense maps in \mathbf{PRO}_m are onto (see Lemma 3.18). So s -dense and p -dense maps are also onto here. Also, it is clear that onto maps are m (respectively, s and p)-dense.

(3) Let $f : A \rightarrow B$ be an s -dense map in \mathbf{PRO}_s . If $g, h : B \rightarrow C$ are morphisms in \mathbf{PRO}_s such that $hf = gf$, then for $b \in B = C_s(f(A))$ we have $nh(b) = h(nb) = g(nb) = ng(b)$, for every $n \in \mathbb{N}$. Now the fact that C is s -separated, implies $h(b) = g(b)$. So, $h = g$ and f is epic. Conversely, let $f : A \rightarrow B$ be epic. Let $f = h \circ e$ be an (s -dense, s -closed) factorization of f . By Corollary 4.24 in the next section, there is a retraction h' such that $h'h = id$. Hence, $(hh')h = h(h'h) = h$. But h is epic, since f is so. Thus $hh' = id$ and so h is an isomorphism. Then f , being a composition of an s -dense map and an isomorphism, is s -dense. The second part follows from Lemma 3.18. ■

4. Injectivity of projection and separated projection algebras

In this final section, the behaviour of injectivity with respect to C -dense (C -closed) monomorphisms, for $C = C_m, C_s, C_p$, and ordinary injectivity is investigated. The results extends [8].

First recall the following injectivity definition.

Definition 4.22. A projection algebra A is called *m-dense injective* (*s-dense injective*, *p-dense injective*) if it is injective with respect to m -dense (respectively, s -dense, p -dense) monomorphisms. That is, $\text{Hom}(-, A)$ maps dense monomorphisms in **PRO** to epimorphisms in **Set**.

It is clear that injectivity implies m -injectivity, this implies s -injectivity, and the latter implies p -injectivity.

To study these injectivities, first we recall the following result.

Theorem 4.23. [8] *A projection algebra A is a retract of its extension B if and only if $C_p(A) = C_s(A)$.*

Proof. Let A be a retract of B . So, there is a projection map $f : B \rightarrow A$ such that $f|_A = \text{id}_A$. By Lemma 2.9, $C_p(A) \subseteq C_s(A)$. Let $b \in C_s(A)$. Then, $nb \in A$, for all $n \in \mathbb{N}$. So, $nf(b) = f(nb) = nb$, for all $n \in \mathbb{N}$. Since $f(b) \in A$, this shows that $b \in C_p(A)$. Conversely, let $C_p(A) = C_s(A)$. Define $g : B \rightarrow A$ by $g(b) = b$, for $b \in A$, and $g(b) = a$, for $b \in C_s(A) \setminus A$, where $a \in A$ is chosen in A such that $nb = na$, for all $n \in \mathbb{N}$, which exists since $C_s(A) = C_p(A)$. Also, for $b \in B \setminus C_s(A)$, define $g(b) = a_0$, where a_0 is a zero element of A , if $\mathbb{N} \cap A = \emptyset$, and define $g(b) = (k-1)b$, where k is the least natural number with $kb \notin A$, if $\mathbb{N} \cap A \neq \emptyset$. Then g is a projection map, and $g|_A = \text{id}_A$. ■

Corollary 4.24. *The s -closed (m -closed) one-one projection maps are retractable, but not conversely. Also, the p -closed one-one projection maps are not necessarily retractable*

Proof. Let A, B be projection algebras with $A \leq B$, and A be s -closed in B . Applying Theorem 4.23, we show that $C_s(A) = C_p(A)$. But, $C_s(A) = A \subseteq C_p(A)$, since A is s -closed. And the other inclusion is always true. To see that the converses are not true, consider an injective projection algebra A (for example take $A = \mathbb{N}^\infty$). Using Lemma 3.19, A has a proper s -dense extension, say B . Then, A being injective is a retract of B , but it is not s -closed in B , since otherwise A being s -closed and s -dense in B , is equal to B , a contradiction.

For the last part, consider the inclusion map $\mathbb{N} \hookrightarrow \mathbb{N}^\infty$ (see the proof of Lemma 2.9). ■

The situation for the separated projection algebras is as follows.

Corollary 4.25.

- (1) *In \mathbf{PRO}_s , s -closed monomorphisms are exactly retractable ones.*
- (2) *In \mathbf{PRO}_m , all monomorphisms are retractable.*

Proof. (1) Let A be a retract of its extension B . By Theorem 4.23, $C_s(A) = C_p(A)$. Then for $b \in C_s(A)$, there exists $a \in A$ such that $nb = na$, for all $n \in \mathbb{N}$. Since B is s -separated, this implies that $b = a \in A$. So, A is s -closed in B . The converse is true by Corollary 4.24.

(2) Let $A \leq B$ in \mathbf{PRO}_m . By Lemma 3.18, A is s -closed in B . So, by Corollary 4.24, A is a retract of B . ■

Theorem 4.26. *In \mathbf{PRO}_m , all objects are m (respectively, s and p)-dense injective, as well as injective.*

Proof. Applying the above corollary, all objects in \mathbf{PRO}_m are injective. Also, by Lemma 3.18, m (s or p)-dense monomorphisms in this category are isomorphisms and have inverses. So, all objects are also m (s or p)-dense injective. ■

Also, by Lemma 3.18, the only p -dense monomorphisms in \mathbf{PRO}_s are isomorphisms. So,

Theorem 4.27. *In \mathbf{PRO}_s , all objects are p -dense injective.*

For p -closure, we have

Lemma 4.28. *In \mathbf{PRO} , p -dense monomorphisms are retractable.*

Proof. We apply Theorem 4.23. Let $A \leq B$ be projection algebras and A be p -dense in B . Then $C_p(A) = B$ and so $C_s(A) \subseteq C_p(A)$. The other inclusion always holds. ■

Notice that, the converse of the above lemma does not hold. For example, consider $\downarrow k \hookrightarrow \mathbb{N}$, for $k \neq \infty$.

The above lemma implies that

Theorem 4.29. *In \mathbf{PRO} , all objects are p -dense injective.*

Now we characterize injectivity in \mathbf{PRO}_s .

Theorem 4.30. *For an s -separated projection algebra A , the following are equivalent:*

- (1) *A is m -dense injective in \mathbf{PRO}_s .*
- (2) *A is s -dense injective in \mathbf{PRO}_s .*
- (3) *A is injective in \mathbf{PRO}_s .*
- (4) *A is injective in \mathbf{PRO} .*

Proof. (2) \Rightarrow (3): Consider a monomorphism $h : B \rightarrow C$ and a morphism $f : B \rightarrow A$ in \mathbf{PRO}_s . Let $h = lg : B \rightarrow D \rightarrow C$ be an (s -dense, s -closed) factorization of h . Then, g is monic and since A is s -dense injective, there exists a projection map $g' : D \rightarrow A$ such that $g'g = f$. Also, since l is s -closed, by Corollary 4.25, there exists a projection map $l' : C \rightarrow D$ such that $l'l = id_D$. Now, $g'l' : C \rightarrow A$ is a projection map with $g'l'h = g'l'lg = g'g = f$.

(3) \Rightarrow (4) proved in [14].

The other parts are clearly true. ■

Lemma 4.31. *In \mathbf{PRO} , for a projection algebra A the following are equivalent:*

- (1) A is a retract of each of its m -dense extension.
- (2) A is a retract of each of its s -dense extension.
- (3) A is a retract of each of its extension.

Proof. (1) \Rightarrow (2) is true, because s -dense maps are m -dense.

(2) \Rightarrow (3): Let B be an extension of A . Consider the (s -dense, s -closed) factorization fg of the inclusion map $\iota_A : A \hookrightarrow B$:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \searrow g & \nearrow f \\ & C_s(A) & \end{array}$$

By Corollary 4.25, there exists a retraction $f' : B \rightarrow C_s(A)$, and by (2) there is a retraction $g' : C_s(A) \rightarrow A$. Then $g'f'$ is the required retraction.

(3) \Rightarrow (1) is trivial. ■

Theorem 4.32. *In \mathbf{PRO} , for a projection algebra A the following are equivalent:*

- (1) A is m -dense injective.
- (2) A is s -dense injective.
- (3) A is injective.

Proof. It is enough to prove (2) \Rightarrow (3). Let A be s -dense injective. Then A is a retract of each of its s -dense extension. So, by the above lemma, A is absolute retract. Hence, A is injective (see Proposition 1.2). ■

We close the paper by the following remarks.

Remark 4.33. Notice that, for each closure operator C on an equational category \mathcal{A} of algebras with enough injectives, defining C -injectives as injectives with respect to C -dense monomorphisms, it can be shown, in a similar way as given above, that C -injectivity and injectivity coincide, whenever in \mathcal{A} each morphism f has a (D,R) factorization $f = hg$, where D is the class of C -dense maps and R is the class of retractable monomorphisms.

Remark 4.34. Define a *divisible* (*m -divisible*) projection algebra to be a projection algebra A such that $nA = A$, for all $n \in \mathbb{N}$ ($mA = A$). By Lemma 3.15(2), A is m -divisible if and only if $A \in \mathbf{PRO}_m$. Also, A is divisible if and only if $A \in \mathbf{PRO}_m$, for all $m \in \mathbb{N}$, if and only if $A \in \mathbf{PRO}_1$. Moreover, in \mathbf{PRO} , m -divisibility implies injectivity, and in \mathbf{PRO}_m , m -divisibility and injectivity are equivalent.

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