ON EIGHT ORDER MOCK THETA FUNCTION

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Abstract. In the present paper we have introduced partial mock theta functions of orders eight. We have established certain identities relating these functions with those of different orders.

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1. Introduction

The discovery of mock theta functions, the last gift of Ramanujan to the world of mathematics, was communicated by him in his last letter to Hardy [13, 14]. Ramanujan wrote that he has discovered some very interesting functions, called them as mock theta functions, defined by a q-series and which for $q \to e^{2\pi ir/s}$ along some radius vector of the unit circle $|q| = 1$, has precisely same behavior as that of one of Jacobi's theta functions. He gave a list of seventeen such functions classifying them as of order three, five and seven, but did not say what he meant by order. So far there is no widely acceptable definition of the order, although, several definitions of the order have been given [1, 11]. The recent work of Bringmann and Ono [4, 5, 6] also clarifies the concept of order. Further, after the discovery of Ramanujan's 'Lost' Notebook more mock theta functions were identified and studied by Andrews and Hickerson [6] and Choi [9] who have designated them as mock theta functions of order six and ten respectively. Recently, Gordon and McIntosh [10] have defined the following eighth mock theta functions of order eight

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \qquad S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \qquad T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \qquad U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \qquad V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

where $(a;q^k)_n$ is q-shifted factorial, defined as

$$\left(a;q^k\right)_n=\prod_{r=0}^{n-1}(1-aq^{kr}),\,(a;q^k)_0=1,$$

Later, Gordon and McIntosh [11] gave a definition of the order of mock theta function and in [12], the mock theta functions $U_0(q)$, $U_1(q)$, $V_0(q)$ and $V_1(q)$ have been reclassified as of second order. In the last one decade the truncated (finite) series of mock theta functions have been studied by several mathematicians [2, 7, 8] who have established the identities connecting functions of different orders. In the present paper, we have considered the partial series of eighth order mock theta functions, i.e. the series of first $m+1$ terms, and have designated them as partial mock theta functions of order eight. Therefore, an eighth order partial mock theta function corresponding to $S_0(q)$ is

$$S_{0m}(q)=\sum_{n=0}^{m}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}.$$

We have established certain identities connecting these function with following mock theta function of various other orders.

Third Order [16]

$$\Phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n} \qquad \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

Fifth Order [16]

$$\Psi_0(q)=\sum_{n=0}^{\infty}q^{(n+1)(n+2)/2}(-q)_n \qquad \Psi_1(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}(-q)_n$$

$$\varphi_0(q)=\sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n \qquad \varphi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}(-q;q^2)_n$$

Sixth Order [3]

$$\varphi(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2}(q;q^2)_n}{(-q)_{2n}} \qquad \psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^nq^{(n+1)^2}(q;q^2)_n}{(-q)_{2n+1}}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q)_n}{(q;q^2)_{(n+1)}} \qquad X(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q)_n}{(q;q^2)_{(n+1)}}$$

2. Main Results

In this section we give the following identities, which show the relations of eighth order (partial) mock theta functions and mock theta functions of various other orders.

$$(-q;q^2))_\infty \sum_{m=0}^{\infty} (-q^3;q^2)_m^{-1} q^{2m} S_{0m}(q) = (-q;q^2)_\infty \Phi(q) - S_0(q) \tag{2.1}$$

$$q^2 \sum_{m=0}^{\infty} (-q;q^2)_m q^{2m} S_{0m}(q) = (-q^2;q^2)_\infty S_0(q) - \varphi_0(q) \tag{2.2}$$

$$(q;q^2))_\infty \sum_{m=0}^{\infty} (q^3;q^2)_m^{-1} q^{2m} S_{0m}(q) = (q;q^2)_\infty \varphi(-q) - S_0(q) \tag{2.3}$$

$$q^2(1+q)(q^3;q^2)_\infty \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2m}}{(q^3;q^2)_m} S_{0m}(q) = (q^2;q^2)_\infty S_0(q) - (q;q^2)_\infty \overline{V}_0(q) \tag{2.4}$$

where $\overline{V}_q = (1 + V_0(q))/2$.

$$q^3 \sum_{m=0}^{\infty} (-q^2;q^2)_m q^{2m} S_{1m}(q) = \varphi_1(q) - q(-q^2;q^2)_\infty S_1(q) \tag{2.5}$$

$$q^4(q^5;q^2)_\infty \sum_{m=0}^{\infty} (q^5;q^2)_m^{-1} q^{2m} S_{1m}(q) = (q^3;q^2)_\infty \psi(-q) + S_1(q) \tag{2.6}$$

$$q^2(q;q^2)_\infty \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{2m}}{(q^3;q^2)_m} S_{1m}(q) = q(q^2;q^2)_\infty S_1(q) - (q;q^2)_\infty V_1(q) \tag{2.7}$$

$$q^3(1+q) \sum_{m=0}^{\infty} (-q^3;q^2)_m q^{2m} T_{0m}(q) = (-q;q^2)_\infty T_0(q) - \Psi_0(q^2) \tag{2.8}$$

$$q^3(q^5;q^2)_\infty \sum_{m=0}^{\infty} (q^5;q^2)_m^{-1} q^{2m} T_{0m}(q) = T_0(q) - (q^3;q^2)_\infty \chi(q^2) \tag{2.9}$$

$$q^2(-q^4;q^2)_\infty \sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1} q^{2m} T_{1m}(q) = T_1(q) - (-q^2;q^2)_\infty \nu(q) \tag{2.10}$$

$$q^3(1+q)\sum_{m=0}^{\infty}(-q^3;q^2)_m q^{2m} T_{1m}(q) = (-q;q^2)_\infty T_1(q) - \Psi_1(q^2) \tag{2.11}$$

$$q^3(q^5;q^2)_\infty \sum_{m=0}^{\infty}(q^5;q^2)_m^{-1} q^{2m} T_{1m}(q) = T_1(q) - (q^3;q^2)_\infty X(q^2) \tag{2.12}$$

$$q\sum_{m=0}^{\infty}(q;q^2)_m q^{2m} V_{0m}(q) = \varphi_0(q) - (q;q^2)_\infty V_0(q) \tag{2.13}$$

$$q^2(-q^4;q^2)_\infty \sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1} q^{2m} V_{0m}(q) = (-q^2;q^2)_\infty \varphi(-q) - V_0(q) \tag{2.14}$$

$$q(1+q)(-q^4;q^2)_\infty \sum_{m=0}^{\infty}(q;q^2)(-q^4;q^2)_m^{-1} q^{2m} V_{0m}(q) = (q;q^2)_\infty V_0(q) + S_0(q) \tag{2.15}$$

$$q^3\sum_{m=0}^{\infty}(q^3;q^2)_m q^{2m} V_{1m}(q) = \varphi_1(q) - (q^3;q^2)_\infty V_1(q) \tag{2.16}$$

$$q^2(-q^4;q^2)_\infty \sum_{m=0}^{\infty}(-q^4;q^2)_m^{-1} q^{2m} V_{1m}(q) = (-q^2;q^2)_\infty \psi(-q) - V_1(q) \tag{2.17}$$

$$(1-q^2)\sum_{m=0}^{\infty} q^{2m} S_{0m}(q) = S_1(q) \tag{2.18}$$

$$q^2(1-q^2)\sum_{m=0}^{\infty} q^{2m} T_{1m}(q) = T_0(q) \tag{2.19}$$

Proof. To prove identities (2.1) - (2.17), we consider a simple series identity, subject to the convergence of the series involved

$$A(q)\sum_{m=0}^{\infty} B_m(q)\sum_{r=0}^{m}\alpha_r + C_\infty(q)\sum_{m=0}^{\infty}\alpha_m = \sum_{m=0}^{\infty} C_m(q)\alpha_m, \tag{2.20}$$

where

$$A(q) = \frac{(aq-e)(e-bq)}{(q-e)(e-abq)},\ B_m(q) = \frac{(a,b;q)_m(q)^m}{(e,abq^2/e;q)_m},\ C_m(q) = \frac{(a,b;q)_m}{(e/q,abq/e;q)_m}.$$

The Series identity (2.20) can easily be proved [1] by using the summation formula

$${}_3\Phi_2(a,b,q;e,f;q)_n = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1 - \frac{(a,b;q)_{n+1}}{(e/q,abq/e;q)_{n+1}}\right],$$

for $ef = abq^2$ which is a particular case of a transformation between two Saalschutzian terminating ${}_3\Phi_2(..)$ due to Sears [15].

Taking $\alpha_r = \dfrac{q^{r(r+2)}(-q;q^2)_r}{(-q^2;q^2)_r}$, $a=0,\ b=-q^2$ and then $e=0$, we get (2.1).

The identities (2.2) - (2.17) can now easily be proved with a proper choice of the sequence α_r and the parameters $a,\ b$ and e in (2.20).

To prove (2.18) and (2.19), we consider the identity

$$\sum_{m=0}^{p}\beta_m\sum_{r=0}^{m}\alpha_r = \sum_{r=0}^{p}\alpha_r\sum_{m=0}^{p}\beta_m - \sum_{r=1}^{p}\alpha_r\sum_{m=0}^{r-1}\beta_m. \tag{2.21}$$

Taking $\beta_m = q^{\lambda m}$, after some simplification we get

$$\sum_{r=0}^{\infty}q^{\lambda r}\sum_{m=0}^{r}\alpha_m q^m = (1-q^\lambda)^{-1}\sum_{m=0}^{\infty}\alpha_m(q)^{m(1+\lambda)}. \tag{2.22}$$

Now for $\alpha_m = \dfrac{q^{m^2-m}(-q;q^2)_m}{(-q^2;q^2)_m}$, $\lambda = 2$ and for $\alpha_m = \dfrac{q^{m^2}(-q^2;q^2)_m}{(-q;q^2)_m}$, $\lambda = 2$, (2.22) gives (2.18) and (2.19) respectively.

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