ON THE QUALITATIVE BEHAVIORS OF SOLUTIONS TO A KIND OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

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Abstract. In this paper, with use of a Lyapunov functional, we discuss stability and boundedness of solutions to a kind of nonlinear third order differential equation with retarded argument:

$$\begin{aligned} &x'''(t) + h(x(t), x'(t), x''(t), x(t-r(t)), x'(t-r(t)), x''(t-r(t)))x''(t) \\ &\quad + g(x(t-r(t)), x'(t-r(t))) + f(x(t-r(t))) \\ &\quad = p(t, x(t), x'(t), x(t-r(t)), x'(t-r(t)), x''(t)), \end{aligned}$$

when $p(t, x(t), x'(t), x(t-r(t)), x'(t-r(t)), x''(t)) = 0$ and $\neq 0$, respectively. Our results include and improve some well-known results in the literature. An example is also given to illustrate the importance of results obtained and the topic.

Keywords: stability, boundedness, Lyapunov functional, nonlinear third order differential equations, retarded argument.

AMS (MOS) Subject Classification: 34K20.

1. Introduction

In a recent paper, Afuwape and Omeike [1] discussed stability and boundedness of solutions to nonlinear third order delay differential equation:

$$\begin{aligned} &x'''(t) + h(x'(t))x''(t) + g(x(t-r(t)), x'(t-r(t))) + f(x(t-r(t))) \\ &\quad = p(t, x(t), x'(t), x''(t)), \end{aligned}$$

when $p(t, x(t), x'(t), x''(t)) \equiv 0$ and $\neq 0$, respectively.

In this paper, we consider nonlinear third order differential equation with retarded argument, $r(t)$:

$$
\begin{aligned}
&x'''(t) + h(x(t), x'(t), x''(t), x(t-r(t)), x'(t-r(t)), x''(t-r(t)))x''(t) \\
&\qquad +g(x(t-r(t)), x'(t-r(t))) + f(x(t-r(t))) \\
&\qquad = p(t, x(t), x'(t), x(t-r(t)), x'(t-r(t)), x''(t)),
\end{aligned} \tag{1}
$$

which is equivalent to the system

$$
\begin{aligned}
x'(t) &= y(t), \\
y'(t) &= z(t), \\
z'(t) &= -h(x(t), y(t), z(t), x(t-r(t)), y(t-r(t)), z(t-r(t)))z(t) \\
&\quad -g(x(t), y(t)) - f(x(t)) + \int_{t-r(t)}^{t} g_x(x(s), y(s))y(s)ds \\
&\quad + \int_{t-r(t)}^{t} g_y(x(s), y(s))z(s)ds + \int_{t-r(t)}^{t} f'(x(s))y(s)ds \\
&\quad +p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)),
\end{aligned} \tag{2}
$$

where $0 \le r(t) \le \gamma$, γ is a positive constant which will be determined later, and $r'(t) \le \beta$, $0 < \beta < 1$; the primes in equation (1) denote differentiation with respect to t, $t \in \Re^+$, $\Re^+ = [0, \infty)$; h, g, f and p are continuous functions in their respective arguments on $\Re^6$, $\Re^2$, $\Re$ and $\Re^+ \times \Re^5$, respectively, with $g(x, 0) = f(0) = 0$, in the statement of Theorem 1. The continuity of functions h, g, f and p guarantees the existence of the solution of equation (1) (see [3, pp.14]). In addition, it is also supposed that the derivatives $g_x(x, y) \equiv \frac{\partial}{\partial x} g(x, y)$, $g_y(x, y) \equiv \frac{\partial}{\partial y} g(x, y)$ and $f'(x) \equiv \frac{df}{dx}$ exist and are continuous; all solutions of (1) are real valued and the functions h, g, f and p satisfy a Lipschitz condition in x, y, z, $x(t-r(t))$, $y(t-r(t))$ and $z(t-r(t))$. Then the solution is unique (see [3, pp.14]). Throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively.

The motivation for the present work has been inspired basically by the paper of Afuwape and Omeike [1], Sadek [9] and Tunç ([11], [12]). Our aim here is to extend and improve the results established by Afuwape and Omeike [1] to nonlinear differential equation with retarded argument (1) for the asymptotic stability of trivial solution and boundedness of all solutions of this equation, when $p \equiv 0$ and $p \neq 0$ in (1), respectively. We also give an explanatory example for the illustration of the subject. All aforementioned papers have been published without including an explanatory example on the stability and boundedness of solutions of third order nonlinear differential equations with retarded argument. In addition, to the best of our knowledge, so far throughout all the papers published on the subject of this paper, the second term in (1) has only consisted of $ax''(t)$, $a_1 x''(t)$, $\alpha\, x''(t)$, (a, a_1 and α are constants), $h(x'(t))x''(t)$, $\varphi(x(t), x'(t))x''(t)$, $a(t)x''(t)$, $f(x(t), x'(t), x''(t))x''(t)$ or $a(t)f(x(t), x'(t))x''(t)$ (see [1], [9], [11], [12] and the references registered thereof). But, our second term has the form

$$h(x(t), x'(t), x''(t), x(t-r(t)), x'(t-r(t)), x''(t-r(t)))x''(t).$$

This case, clearly, is an improvement.

2. Preliminaries

Now, we will give some basic information for the general non-autonomous differential system with retarded argument (see, also, the books of Èl'sgol'ts [3], Hale [4], Kolmanovskii and Myshkis [5], Kolmanovskii and Nosov [6], Krasovskii [7] and Yoshizawa [13]). Consider the general non-autonomous differential system with retarded argument:

$$\dot{x} = f(t, x_t), x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0, \tag{3}$$

where $f : [0,\ \infty) \times C_H \rightarrow \Re^n$ is a continuous mapping, $f(t,0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C,\ \|.\|)$ is the Banach space of continuous function $\phi : [-r,\ 0] \rightarrow \Re^n$ with supremum norm, $r > 0$; C_H is the open H-ball in C; $C_H := \{\phi \in (C\,[-r, 0]\,,\ \Re^n) : \|\phi\| < H\}$.

Definition 1. (See [13].) A function $x(t_0, \phi)$ is said to be a solution of (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \geq 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - r,\ \ t_0 + A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0, \phi) \in C_H$ for $t_0 \leq t < t_0 + A$,

(ii) $x_{t_0}(t_0, \phi) = \phi$,

(iii) $x(t_0, \phi)$ satisfies (3) for $t_0 \leq t < t_0 + A$.

Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0,\ t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|\cdot|$ will denote a convenient norm in $\Re^n$ with $|x| = \max\limits_{1 \leq i \leq n} |x_i|$.

Definition 2. (See [2].) A continuous function $W : [0,\ \infty) \rightarrow [0,\ \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 3. (See [2].) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : [0,\ \infty) \times D \rightarrow [0,\ \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.

Definition 4. (See [2].) Let $f(t, 0) = 0$. The zero solution of equation (3) is:

(i) stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$ there exists $\delta > 0$ such that $[\phi \in C(t_1),\ \ \|\phi\| < \delta, t \geq t_1]$ implies that $|x(t, t_1, \phi)| < \varepsilon$.

(ii) asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an $\eta > 0$ such that $[\phi \in C(t_1),\ \ \|\phi\| < \delta]$ implies that $x(t, t_0, \phi) \rightarrow 0$ as $t \rightarrow \infty$.

Definition 5. (See [2].) Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}$ and is defined by the following relation:

$$\dot{V}(t,\phi) = \limsup_{h\to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

For the general autonomous delay differential system

$$\dot{x} = f(x_t), \tag{4}$$

which is a special case of (3), the following lemma is given:

Lemma. (See[10].) *Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $V(\phi) \geq 0$, $\dot{V}(\phi) \leq 0$, then the solution $x_t = 0$ of (4) is stable.*

If we define $Z = \left\{\phi \in C_H : \dot{V}(\phi) = 0\right\}$, then the solution $x_t = 0$ of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. Main result

In this section, we establish two theorems, which are the main results of this paper.

First, for the case

$$p(t,x,y,x(t-r(t)),y(t-r(t)),z) \equiv 0,$$

the following result is introduced.

Theorem 1. *In addition to the basic assumptions imposed on the functions h, g and f that appearing in (1), we assume that there are positive constants a, b, c, ε, ρ, μ, K, L and M such that the following conditions hold:*

(i) $ab - c > 0$.

(ii) $f(x)\operatorname{sgn} x > 0$, $(x \neq 0)$, $\sup\{f'(x)\} = c$, $|f'(x)| \leq L$.

(iii) $\dfrac{g(x,y)}{y} \geq b + \varepsilon$, $(y \neq 0)$, $|g_x(x,y)| \leq K$, $|g_y(x,y)| \leq M$.

(iv) $\rho \leq h(x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t))) - a \leq 2(\varepsilon\rho\mu^{-1})^{\frac{1}{2}}$.

Then the zero solution of equation (1) *is asymptotically stable, provided that*

$$\gamma < \min\left\{\frac{2(\mu b - c)}{\mu(K+L+M)+2\lambda},\ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}$$

with $\mu = \dfrac{ab+c}{2b}$, *where* γ *is the bound on* $r(t)$.

Proof. To verify Theorem 1, we define the following Lyapunov functional $V_1 = V_1(x_t, y_t, z_t)$:

$$\begin{aligned} V_1(x_t,y_t,z_t) = {} & \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{1}{2}\mu a y^2 + \int_0^y g(x,\eta)d\eta + \mu yz \\ & + \frac{1}{2}z^2 + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds, \end{aligned} \tag{5}$$

where λ and δ are positive constants which will be determined later in the proof.

Now, it is obvious that $V_1(0,0,0)=0$.

We also have, by the assumption $\dfrac{g(x,y)}{y} \ge b+\varepsilon$, $(y \ne 0)$,

$$\begin{aligned} V_1(x_t,y_t,z_t) &= \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{1}{2}\mu a y^2 + \int_0^y \frac{g(x,\eta)}{\eta}\eta d\eta + \mu yz + \frac{1}{2}z^2 \\ &\quad + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds \\ &\ge \mu\int_0^x f(\xi)d\xi + yf(x) + \frac{\mu a}{2}y^2 + \frac{b}{2}y^2 + \frac{\varepsilon}{2}y^2 + \mu yz + \frac{1}{2}z^2 \\ &\quad + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds \\ &= \frac{1}{2b}\left[by+f(x)\right]^2 + \mu\int_0^x f(\xi)d\xi + \frac{\mu a}{2}y^2 + \frac{\varepsilon}{2}y^2 - \frac{1}{2b}f^2(x) \\ &\quad + \mu yz + \frac{1}{2}z^2 + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds \\ &= \frac{1}{2by^2}\left[4\int_0^x f(\xi)\left\{\int_0^y (\mu b - f'(\xi))\eta d\eta\right\}d\xi\right] \\ &\quad + \frac{\varepsilon}{2}y^2 + \frac{1}{2}(\mu y + z)^2 + \frac{1}{2}\mu(a-\mu)y^2 + \frac{1}{2b}\left[by+f(x)\right]^2 \\ &\quad + \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds. \end{aligned} \tag{6}$$

By using the assumptions $a-\mu = \dfrac{ab-c}{2b} > 0$ and $\mu b - f'(x) \ge \dfrac{ab-c}{2} > 0$, it follows from (6) that there exist sufficiently small positive constants D_i, $(i = 1,\ 2,\ 3)$, such that

$$
\begin{aligned}
V_1(x_t, y_t, z_t) &\geq D_1x^2 + D_2y^2 + D_3z^2 \\
&+\lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds + \delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds \\
&\geq D_4(x^2 + y^2 + z^2) + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds + \delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds \\
&\geq D_4(x^2 + y^2 + z^2),
\end{aligned} \tag{7}
$$

since the integrals $\int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative, where $D_4 = \min\{D_1, D_2, D_3\}$. Now, we can deduce that there exists a continuous function u with $u(|\phi(0)|) \geq 0$ such that $u(|\phi(0)|) \leq V(\phi)$.

Next, by a straightforward calculation from (5) and (2), we compute the total derivative of $V_1(x_t, y_t, z_t)$ with respect to t :

$$
\begin{aligned}
\frac{d}{dt}V_1(x_t, y_t, z_t) &= f'(x)y^2 + \mu z^2 - \mu y g(x, y) + y \int_0^y g_x(x, \eta)d\eta \\
&-\mu \left\{h(x, y, x(t - r(t)), y(t - r(t)), z(t - r(t))) - a\right\} yz \\
&-h(x, y, x(t - r(t)), y(t - r(t)), z(t - r(t)), z)z^2 \\
&+(\mu y + z) \int_{t-r(t)}^{t} f'(x(s))y(s)ds + (\mu y + z) \int_{t-r(t)}^{t} g_x(x(s), y(s))y(s)ds \\
&+(\mu y + z) \int_{t-r(t)}^{t} g_y(x(s), y(s))z(s)ds + \lambda y^2 r(t) + \delta z^2 r(t) \\
&-\lambda(1 - r'(t)) \int_{t-r(t)}^{t} y^2(s)ds - \delta(1 - r'(t)) \int_{t-r(t)}^{t} z^2(s)ds.
\end{aligned} \tag{8}
$$

By use of the assumptions of Theorem 1 and the inequality $2|uv| \leq u^2 + v^2$, we obtain

$$
-h(x, y, x(t - r(t)), y(t - r(t)), z(t - r(t)), z)z^2 \leq -(a + \rho)z^2, \tag{9}
$$

$$
-\left(\mu \frac{g(x, y)}{y} - f'(x)\right) y^2 \leq -(\mu b + \mu\varepsilon - c)y^2, \tag{10}
$$

$$
\begin{aligned}
\mu y \int_{t-r(t)}^{t} f'(x(s))y(s)ds &\leq \frac{\mu L r(t)}{2} y^2 + \frac{\mu L}{2} \int_{t-r(t)}^{t} y^2(s)ds \\
&\leq \frac{\mu L\gamma}{2} y^2 + \frac{\mu L}{2} \int_{t-r(t)}^{t} y^2(s)ds,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
z \int_{t-r(t)}^{t} f'(x(s))y(s)ds &\leq \frac{L r(t)}{2} z^2 + \frac{L}{2} \int_{t-r(t)}^{t} y^2(s)ds \\
&\leq \frac{L\gamma}{2} z^2 + \frac{L}{2} \int_{t-r(t)}^{t} y^2(s)ds,
\end{aligned} \tag{12}
$$

$$(13)\quad \begin{aligned} \mu y\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\leq \frac{\mu K r(t)}{2}y^2+\frac{\mu K}{2}\int_{t-r(t)}^{t} y^2(s)ds \\ &\leq \frac{\mu K\gamma}{2}y^2+\frac{\mu K}{2}\int_{t-r(t)}^{t} y^2(s)ds, \end{aligned}$$

$$(14)\quad \begin{aligned} z\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)ds &\leq \frac{K r(t)}{2}z^2+\frac{K}{2}\int_{t-r(t)}^{t} y^2(s)ds \\ &\leq \frac{K\gamma}{2}z^2+\frac{K}{2}\int_{t-r(t)}^{t} y^2(s)ds, \end{aligned}$$

$$(15)\quad \begin{aligned} \mu y\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\leq \frac{\mu M r(t)}{2}y^2+\frac{\mu M}{2}\int_{t-r(t)}^{t} z^2(s)ds \\ &\leq \frac{\mu M\gamma}{2}y^2+\frac{\mu M}{2}\int_{t-r(t)}^{t} z^2(s)ds, \end{aligned}$$

$$(16)\quad \begin{aligned} z\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)ds &\leq \frac{M r(t)}{2}z^2+\frac{M}{2}\int_{t-r(t)}^{t} z^2(s)ds \\ &\leq \frac{M\gamma}{2}z^2+\frac{M}{2}\int_{t-r(t)}^{t} z^2(s)ds, \end{aligned}$$

$$(17)\quad \lambda y^2 r(t)\leq \lambda\gamma y^2,\delta z^2 r(t)\leq \delta\gamma z^2.$$

Combining the inequalities (9)-(17) into (8), we obtain

$$(18)\quad \begin{aligned} &\frac{d}{dt}V_1(x_t,y_t,z_t)-\left(\mu b-c-\frac{\mu K}{2}\gamma-\frac{\mu L}{2}\gamma-\frac{\mu M}{2}\gamma-\lambda\gamma\right)y^2 \\ &-\left(a-\mu-\frac{K}{2}\gamma-\frac{L}{2}\gamma-\frac{M}{2}\gamma-\delta\gamma\right)z^2 \\ &-(\mu\varepsilon)y^2-\mu\left\{h(x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\right\}yz \\ &-\rho z^2+\left[\frac{K}{2}+\frac{L}{2}+\frac{\mu K}{2}+\frac{\mu L}{2}-(1-\beta)\lambda\right]\int_{t-r(t)}^{t} y^2(s)ds \\ &+\left[\frac{M}{2}+\frac{\mu M}{2}-(1-\beta)\delta\right]\int_{t-r(t)}^{t} z^2(s)ds. \end{aligned}$$

Now, we consider the terms

$$W=:(\mu\varepsilon)y^2+\mu\left\{h(x,y,z,x(t-r(t)),y(t-r(t)),z(t-r(t)))-a\right\}yz+\rho z^2,$$

which are contained in (18). Clearly, by the assumption (iv), we get

$$\begin{aligned} W &\geq (\mu\varepsilon)y^2-2\mu(\varepsilon\rho\mu^{-1})^{\frac{1}{2}}\,|y|\;\;|z|+\rho z^2 \\ &=(\mu\varepsilon)y^2-2(\varepsilon\rho\mu)^{\frac{1}{2}}\,|y|\;\;|z|+\rho z^2 \\ &=[\sqrt{\mu\varepsilon}\;\;|y|-\sqrt{\rho}\;\;|z|]^2\geq 0. \end{aligned}$$

This estimate implies that

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t) &\le -\left(\mu b - c - \frac{\mu K}{2}\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\ &- \left(a - \mu - \frac{K}{2}\gamma - \frac{L}{2}\gamma - \frac{M}{2}\gamma - \delta\gamma\right) z^2 \\ &+ \left[\frac{K}{2} + \frac{L}{2} + \frac{\mu K}{2} + \frac{\mu L}{2} - (1-\beta)\lambda\right] \int_{t-r(t)}^{t} y^2(s)ds \\ &+ \left[\frac{M}{2} + \frac{\mu M}{2} - (1-\beta)\delta\right] \int_{t-r(t)}^{t} z^2(s)ds. \end{aligned} \tag{19}$$

By taking $\lambda = \frac{1}{2(1-\beta)}(K+L)(1+\mu)$ and $\delta = \frac{1}{2(1-\beta)}M(1+\mu)$, we get from (19) that

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t) &\le -\left(\mu b - c - \frac{\mu K}{2}\gamma - \frac{\mu L}{2}\gamma - \frac{\mu M}{2}\gamma - \lambda\gamma\right) y^2 \\ &- \left(a - \mu - \frac{K}{2}\gamma - \frac{L}{2}\gamma - \frac{M}{2}\gamma - \delta\gamma\right) z^2. \end{aligned} \tag{20}$$

The above inequity, that is, (20), yields

$$\frac{d}{dt}V_1(x_t,y_t,z_t) \le -k_1 y^2 - k_2 z^2 \le 0 \tag{21}$$

for some positive constants k_1 and k_2 provided that

$$\gamma < \min\left\{ \frac{2(\mu b - c)}{\mu(K+L+M) + 2\lambda}, \ \frac{2(a-\mu)}{K+L+M+2\delta} \right\}.$$

It is also clear that the largest invariant set in Z is $Q = \{0\}$, where

$$Z = \left\{ \phi \in C_H : \dot{V}_1(\phi) = 0 \right\}.$$

Namely, the only solution of equation (1) for which $\frac{d}{dt}V_1(x_t,y_t,z_t) = 0$ is the solution $x_t \equiv 0$. Thus, under the above discussion, we conclude that the trivial solution of equation (1) is asymptotically stable. This fact completes the proof of Theorem 1.

In the case

$$p(t,x,y,x(t-r(t)),y(t-r(t)),z) \neq 0,$$

we establish the following result.

Theorem 2. *We assume that the assumptions* (i)-(iv) *of Theorem* 1 *and the following condition hold:*

$$|p(t,x,y,x(t-r(t)),y(t-r(t)),z)| \le q(t),$$

where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions. Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t)=\phi(t), x'(t)=\phi'(t), x''(t)=\phi''(t)$$

satisfies the inequalities

$$|x(t)|\le\sqrt{K},\ |x'(t)|\le\sqrt{K},\ |x''(t)|\le\sqrt{K}$$

for all $t\ge t_0$, where $\phi\in C^2\left([t_0-r,t_0],\ \Re\right)$, provided that

$$\gamma<\min\left\{\frac{2(\mu b-c)}{\mu(K+L+M)+2\lambda},\ \frac{2(a-\mu)}{K+L+M+2\delta}\right\}$$

with $\mu=\dfrac{ab+c}{2b}$.

Proof. To prove Theorem 2, we use the Lyapunov functional, $V_1=V_1(x_t,y_t,z_t)$, defined by (5). Now, taking into account the assumptions of Theorem 2, the result of Theorem 1 and (21), a straightforward calculation from (5) and (2) gives that

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le -k_1y^2-k_2z^2+(\mu y+z)p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t)).$$

Hence

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t)&\le(\mu\,|y|+|z|)\,|p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t))|\\&\le(\mu\,|y|+|z|)q(t)\le D_5(|y|+|z|)q(t),\end{aligned}$$

where $D_5=\max\{1,\ \mu\}$.

Now, by using the inequalities $|y|<1+y^2$ and $|z|<1+z^2$, we have

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le D_5(2+y^2+z^2)q(t).$$

The estimate (7) implies that

$$y^2+z^2\le D_4^{-1}\,V_1(x_t,y_t,z_t).$$

The last two inequalities lead to

$$\begin{aligned}\frac{d}{dt}V_1(x_t,y_t,z_t)&\le D_5(2+D_4^{-1}V_1(x_t,y_t,z_t))q(t)\\&=2D_5q(t)+D_5D_4^{-1}V_1(x_t,y_t,z_t)q(t).\end{aligned}\tag{22}$$

Now, we integrate (22) from 0 to t and use the assumption $q\in L^1(0,\infty)$, and Gronwall-Reid-Bellman inequality to make the next estimate:

$$\begin{aligned}V_1(x_t,y_t,z_t)&\le V_1(x_0,y_0,z_0)+2D_5A+D_5D_4^{-1}\int_0^t(V_1(x_s,y_s,z_s))\,q(s)ds\\&\le(V_1(x_0,y_0,z_0)+2D_5A)\exp\left(D_5D_4^{-1}\int_0^t q(s)ds\right)\\&\le(V_1(x_0,y_0,z_0)+2D_5A)\exp\left(D_5D_4^{-1}A\right)=K_1<\infty,\end{aligned}\tag{23}$$

where $K_1 > 0$ is a constant, $K_1 = (V_1(x_0, y_0, z_0) + 2D_5A)\ \exp\left(D_5D_4^{-1}A\right)$ and $A = \displaystyle\int_0^{\infty} q(s)ds$.

In view of (7) and (23), it follows that

$$x^2 + y^2 + z^2 \leq D_4^{-1}V_1(x_t, y_t, z_t) \leq K,$$

where $K = K_1D_4^{-1}$. Thus, we can deduce

$$|x(t)| \leq \sqrt{K},\ |y(t)| \leq \sqrt{K},\ |z(t)| \leq \sqrt{K},$$

for all $t \geq t_0$. That is,

$$|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},$$

for all $t \geq t_0$. The proof of Theorem 2 is now complete.

Example. We consider the following third order nonlinear differential equation with retarded argument

$$\begin{aligned} x''' &+ \left(4 + \frac{1}{1+x^2+(x')^2+(x'')^2+x^2(t-r(t))+(x'(t-r(t)))^2+(x''(t-r(t)))^2}\right) x'' \\ &+7x'(t-r(t)) + \sin x'(t-r(t)) + x(t-r(t)) + \operatorname{arctg} x(t-r(t)) \\ &= \frac{1}{1+t^2+x^2+(x')^2+(x(t-r(t)))^2+(x'(t-r(t)))^2+(x'')^2}. \end{aligned}$$

This equation can be stated as the following equivalent system:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\left(4 + \frac{1}{1+(x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}\right) z \\ &\quad -(7y + \sin y) - x - \operatorname{arctg} x + \int_{t-r(t)}^{t} (1 + \frac{1}{1+(x(s))^2})y(s)ds \\ &\quad + \int_{t-r(t)}^{t} (7 + \cos y(s))\, z(s)ds \\ &\quad + \frac{1}{1+t^2+x^2+y^2+x^2(t-r(t))+y^2(t-r(t))+z^2}. \end{aligned} \tag{24}$$

So, we have

$$\begin{aligned} &h(x, y, z, x(t-r(t)), y(t-r(t)), z(t-r(t))) \\ &\quad = 4 + \frac{1}{1+x^2+y^2+z^2+x^2(t-r(t))+y^2(t-r(t))+z^2(t-r(t))}, \end{aligned}$$

$$4 \le 4 + \frac{1}{1 + x^2 + y^2 + z^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t - r(t))} \le 5,$$

that is,

$$\begin{aligned}
&4 \le h(x, y, z, x(t - r(t)), y(t - r(t)), z(t - r(t))) \le 5,\\
&g(x, y) = 7y + \sin y, g(x, 0) = 0,\\
&\frac{g(x, y)}{y} = 7 + \frac{\sin y}{y} \ge 6, (y \ne 0, |y| < \pi),\\
&g_y(x, y) = 7 + \cos y, |g_y(x, y)| = |7 + \cos y| \le 8,\\
&f(x) = x + \operatorname{arctg} x, f(0) = 0,\\
&f'(x) = 1 + \frac{1}{1 + x^2}, |f'(x)| \le 2,\\
&p(t, x, y, x(t - r(t)), y(t - r(t)), z)\\
&\qquad = \frac{1}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2},\\
&|p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \le \frac{1}{1 + t^2} = q(t),
\end{aligned}$$

and hence

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1 + s^2} ds = \frac{\pi}{2} < \infty, \text{ that is, } q \in L^1(0, \infty).$$

Clearly, in cases of appropriate choice of the constants a, b and c, one can easily show that all the assumptions Theorem 1 and Theorem 2 hold for (24). That is, for the case $p(t, x, y, x(t-r(t)), y(t-r(t)), z) = 0$, the trivial solution of our equation is asymptotically stable, and for the case $p(t, x, y, x(t - r(t)), y(t - r(t)), z) \ne 0$, all solutions of the equation are bounded.

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