ON THE HYPERBANACH SPACES

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Abstract. In this paper we are going to define hyperBanach spaces and prove some interesting theorems such as open mapping theorem, closed graph theorem, and uniform boundedness principal in these spaces. Also we define a quasinorm over hypervector spaces that converts a factor hypervector space into a normed hyper vector space.

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1. Introduction

In 1934 Marty [3] introduced a new mathematical structure as a generalization of groups and called it hypergroup. Subsequently, many authors worked on this new field and constructed some other generalizations such as hyperrings, hypermodules, and hyperfields. In 1988 the notion of hypervector space was given by Tallini [11]. She studied some algebraic properties of this new structure in [8], [9], and [10]. A wealth of applications of these new constructions in geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. can be found in [2]. Recently, we studied hypervector spaces in the viewpoint of analysis and generalized some definitions and proved many interesting theorems about them in [5], [6], and [7]. In this paper we are going to define hyperBanach space and prove open mapping theorem, closed graph theorem, and uniform boundedness principal which are very important and have key roles in

Banach space theory for this new space. Also we define a quasinorm over hypervector spaces that converts a factor hypervector space into a normed hyper vector space.

Let $P(X)$ be the power set of a set X, $P^\star(X) = P(X)\backslash\{\varnothing\}$, and K a field. A *hypervector space* over K that is defined in [8], is a quadruplet $(X, +, \circ, K)$ such that $(X, +)$ is an abelian group and

$$\circ : K \times X \longrightarrow P^\star(X)$$

is a mapping that for all $a,\, b \in K$ and $x,\, y \in X$ the following properties holds:

(i) $(a + b) \circ x \subseteq (a \circ x) + (b \circ x)$,

(ii) $a \circ (x + y) \subseteq (a \circ x) + (a \circ y)$,

(iii) $a \circ (b \circ x) = (ab) \circ x$, where $a \circ (b \circ x) = \{\, a \circ y \;:\; y \in b \circ x \,\}$,

(iv) $(-a) \circ x = a \circ (-x)$

(v) $x \in 1 \circ x$.

Note that every vector space is a hypervector space and specially, every field is a hypervector space over itself.

A non-empty subset of a hypervector space X over a field K is called a *subspace* of X if the following holds:

(i) $H - H \subseteq H$,

(ii) $a \circ H \subseteq H$, for every $a \in K$.

Note that $a \circ A = \bigcup_{x \in A} a \circ x$, for every $k \in K$ and $A \subseteq X$.

If H is a subspace of X, the *factor hypervector space* of X with respect to H that is defined in [8], is denoted by $(X/Y,\, +,\, *,\, K)$ and is a hypervector space with the elements

$$\{\, [x] = x + Y \;:\; x \in X \,\},$$

and for every $a \in K$,

$$a * [x] = [a \circ x] = \{\, [y] \;:\; y \in a \circ x \,\}.$$

Let $(X, +, \circ, K)$ be a hypervector space, where K is a valued field. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A *pseudonorm* on X that is defined in [9], is a mapping

$$||\cdot|| : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and $x,\, y \in X$ has the following properties:

(i) $||0|| = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| = |a|\, ||x||$.

A pseudonorm on X is called a *norm*, if:

$$||x|| = 0 \iff x = 0.$$

Let $(X, +_1, \circ_1, K)$ and $(Y, +_2, \circ_2, K)$ be two hypervector spaces. A *strong homomorphism* between X and Y is a mapping

$$f : X \longrightarrow Y$$

such that for all $a \in K$ and x, $y \in X$ the following hold:

(i) $f(x +_1 y) = f(x) +_2 f(y)$,

(ii) $f(a \circ_1 x) = a \circ_2 f(x)$.

A strong homomorphism $f : X \longrightarrow Y$, where $X = (X, +_1, \circ_1, ||\ .\ ||_1, K)$ and $Y = (Y, +_2, \circ_2, ||\ .\ ||_2, K)$ are two normed hypervector spaces is called *bounded* if there exists $M \geqslant 0$ such that $||f(x)||_2 \leqslant M||x||_1$, for every $x \in X$.

As we define in [7], a subset A of X is called *convex* if $t \circ x + (1-t) \circ y \subseteq A$, for every x, $y \in A$ and $0 \leqslant t \leqslant 1$. If $kA \subseteq A$, for every k, $|k| \leqslant 1$, then A is called *balanced*. Also the set A is *absorbing* if for each $x \in X$, there is a positive number s_x, such that $x \in t \circ A$ whenever $t > s_x$.

Let $(X, +, \circ, ||\ .\ ||, K)$ be a normed hypervector space. For $x \in X$ and $\epsilon > 0$ the *open ball* $B_\epsilon(x)$ is defined as

$$B_\epsilon(x) = \{\, y \in X \ :\ ||x - y|| < \epsilon \,\},$$

and the unit ball is the open ball with radius equals to 1. Furthermore, the *closed ball*, $C_\epsilon(x)$ is defined as

$$C_\epsilon(x) = \{\, y \in X \ :\ ||x - y|| \leqslant \epsilon \,\}.$$

The $\{\, B_\epsilon(x) \ :\ x \in X, \epsilon > 0 \,\}$ is a basis for a topology on X which is the topology induced by this norm. The set of all interior points of A is denoted by A°. Also the closure of A denoted by $\overline{A}$.

As we define in [5], a sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $||x_n - x_m|| < \epsilon$, for every m, $n \geqslant N$.

2. HyperBanach spaces

Throughout this section K will denote either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$.

Definition 2.1. A normed hypervector space $X = (X, +, \circ, ||\cdot||, K)$ is called a hyperBanach space if every Cauchy sequence in X is convergent.

Example 2.2. Consider the following hypervector space that is defined in [8]:

Let $(\mathbb{R}^n, +)$ be the classical additive group over $\mathbb{R}^n$ and for every $a \in \mathbb{R}$ let

$$a \circ x = \{\, tax \,:\, 0 \leqslant t \leqslant 1\,\},$$

where tax is the classical multiplication of $\mathbb{R}$ over $\mathbb{R}^n$.

Now, let $||x||$ be the distance of x from the origin in $\mathbb{R}^n$. Then it is easily seen that $(\mathbb{R}^n, +, \circ, ||\cdot||, \mathbb{R})$ is a hyperBanach space.

Theorem 2.3. *Let $X = (X, +, \circ, ||\,.\,||, K)$ be a hyperBanach space such that $k \circ 0 = \{0\}$, for every $k \in K$. Then every closed, convex, and absorbing subset of X includes a neighborhood of the origin.*

Proof. Let C be a closed, convex, and absorbing subset of a hyperBanach space X and let $D = C \cap (-C)$, where $-C$ denotes $\{\,-x : x \in C\}$. It is enough to show that D includes a neighborhood of the origin. If A is a non-empty subset of D, then we have

$$0 \in \frac{1}{2} \circ (A - A) \subseteq \frac{1}{2} \circ A + \frac{1}{2} \circ (-A) \subseteq \frac{1}{2} \circ D + \frac{1}{2} \circ (-D) = \frac{1}{2} \circ D + \frac{1}{2} \circ D \subseteq D,$$

because D is convex. Since the neighborhood $\frac{1}{2} \circ D^\circ + \frac{1}{2} \circ (-D^\circ)$ of the origin must be included in D, so it is enough to prove that $D^\circ \neq \varnothing$.

By contradiction, suppose that $D^\circ = \varnothing$. For each $n \in \mathbb{N}$, the set nD is closed and has empty interior, where

$$nD = \underbrace{D + \cdots + D}_{n \text{ times}},$$

and so $X \setminus nD$ is an open set that is dense in X. Suppose that B_1 is a closed ball in $X \setminus D$ with radius no more than 1. Since $(X \setminus 2D) \cap B_1^\circ$ is a non-empty open set, there is a closed ball B_2 in $B_1 \setminus 2D$ with radius no more than $\frac{1}{2}$. There is a closed ball B_3 in $B_2 \setminus 3D$ with radius no more than $\frac{1}{3}$. Continuing in the obvious way, we find a sequence $\{B_n\}$ of closed balls such that for every $n \in \mathbb{N}$, $B_n \cap nD = \varnothing$, the radius of B_n is no more than $\frac{1}{n}$, and $B_m \subseteq B_n$ if $n \leqslant m$. It follows that the centers of the balls form a Cauchy sequence whose limit x is in each of the balls and hence is in $X \setminus nD$, for every n. Since C is absorbing, there is a positive real number s such that if $t > s$ then x, $-x \in t \circ C$ and therefore $x \in t \circ D$. It implies that $x \in nD$, for some $n \in \mathbb{N}$, a contradiction. This proves the theorem. ∎

Definition 2.4. Let $X = (X, +, \circ, K)$ be a hypervector space. A prenorm on X is a positive real valued function p on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(i) $p(0) = 0$,

(ii) $\sup p(\alpha \circ x) \leqslant |\alpha| p(x)$,

(iii) $p(x + y) \leqslant p(x) + p(y)$,

(iv) $p(x - y) = p(y - x)$.

Definition 2.5. A function f from a normed hypervector space X into the non-negative reals is countably subadditive if

$$f\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} f(x_n),$$

for each convergent series $\sum_{n=1}^{\infty} x_n$ in X.

Theorem 2.6. *Let $X = (X, +, \circ, ||\ .\ ||, K)$ be a hyperBanach space such that $k \circ 0 = \{0\}$, for every $k \in K$, $y \in k^{-1} \circ x$, for $x \in k \circ y$, $0 \neq k \in K$, and $x, y \in X$, and $k \circ \overline{A} \subseteq \overline{k \circ A}$, for every $k \in K$, , and $A \subseteq X$. Then every countably subadditive prenorm on X is continuous.*

Proof. Let p be a countably prenorm on X. Suppose that $x, y \in X$. Then $p(x) \leqslant p(x - y) + p(y)$, and $p(y) \leqslant p(y - x) + p(x) = p(x - y) + p(x)$. So

$$|p(x) - p(y)| = \max\{p(x) - p(y), p(y) - p(x)\} \leqslant p(x - y) = |p(x - y) - p(0)|.$$

Therefore if p is continuous at 0 and x is an element of X, then p is continuous at x, too. Thus it is enough to show that p is continuous at 0.

Let $G = \{x : x \in X,\ p(x) < 1\}$. If $t > 0$, then

$$t \circ G = \bigcup_{x \in G} t \circ x \subseteq \{x : x \in X,\ p(x) < t\}.$$

Thus G is absorbing. If $x, y \in G$ and $0 \leqslant t \leqslant 1$, then

$$\sup p(t \circ x + (1 - t) \circ y) \leqslant \sup p(t \circ x) + \sup p((1 - t) \circ y) \leqslant tp(x) + (1 - t)p(y) < 1,$$

so G is convex. Therefore $\overline{G}$ is a closed, convex, and absorbing subset of X and by Theorem 2.3, it includes an open ball U centered at 0 with some positive radius ϵ. Suppose that there is a positive real number s such that $p(x) < s$ whenever $||x|| < \epsilon$. Now, if $x \in X$ is such that $||X|| < s^{-1} t\epsilon$, then $\sup ||(st^{-1}) \circ x|| < \epsilon$. It shows that for every $y \in (st^{-1}) \circ x$, $p(y) < s$, and therefore $\sup p((s^{-1}t) \circ y) \leqslant (s^{-1}t)p(y) < t$, and so $p(x) < t$. It implies the continuity of p at 0. Thus to complete the proof it is enough to show that such an s exists.

Fix an x in X such that $||x|| < \epsilon$. Since $x \in U \subseteq \overline{G}$, there is $x_1 \in G$ such that $||x - x_1|| < 2^{-1}\epsilon$. Since

$$x - x_1 \in 2^{-1} \circ U \subseteq 2^{-1} \circ \overline{G} \subseteq \overline{2^{-1} \circ G},$$

there is $x_2 \in 2^{-1} \circ G$ such that $||x - x_1 - x_2|| < 2^{-2}\epsilon$. Similarly, there is $x_3 \in 2^{-2} \circ G$ such that $||x - x_1 - x_2 - x_3|| < 2^{-3}\epsilon$. Continuing in this way, we find a sequence $\{x_n\}$ such that $x_n \in 2^{-n+1} \circ G$ and $\left\| x - \sum_{i=1}^{n} x_i \right\| < 2^{-n}\epsilon$, for every $n \in \mathbb{N}$. It follows that $p(x_n) < 2^{-n+1}$, for every $n \in \mathbb{N}$, and $x = \sum_{n=1}^{\infty} x_n$, and so the countable subadditivity of p implies that

$$p(x) = p\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} p(x_n) < 2.$$

Put $s = 2$ and the proof is complete. ■

Example 2.7. In Example 2.2 it is not hard to see that $a \circ 0 = \{0\}$, for every $a \in \mathbb{R}$, $y \in a^{-1} \circ x$, for $x \in a \circ y$ whenever $x, y \in \mathbb{R}^n$ and $0 \neq a \in \mathbb{R}$. Now, we show that $\alpha \circ \overline{A} \subseteq \overline{\alpha \circ A}$, for every $\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$. Suppose that $\alpha \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$ are arbitrary. Let $x \in \alpha \circ \overline{A}$. So there is $y \in \overline{A}$ such that $x \in \alpha \circ y$. It means that there is t_0, $0 \leqslant t_0 \leqslant 1$, such that $x = \alpha t_0 y$. Let $r_0 > 0$ be arbitrary. Since $y \in \overline{A}$, then there is $a \in A$ such that

$$||y - a|| < \frac{r_0}{|\alpha||t_0|}.$$

Put $z = \alpha t_0 a \in \alpha \circ A$. We have

$$||x - z|| = ||\alpha t_0 y - \alpha t_0 a|| = |\alpha t_0|||y - a|| < r_0.$$

Therefore $z \in B_{r_0}(x) \cap \alpha \circ A$, and $x \in \overline{\alpha \circ A}$. So $(\mathbb{R}^n, +, \circ, || \cdot ||, \mathbb{R})$ is a normed hypervector space satisfying the hypothesis of Theorem 2.6.

We say the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} ||x_n||$ is a convergent series. The following lemma can be proved similar as the normed vector spaces. So we omit its proof.

Lemma 2.8. *Let $X = (X, +, \circ, || \cdot ||, K)$ be a normed hypervector space. Then X is a hyperBanach space if and only if every absolutely convergent series in X is convergent.*

A function f from a topological space X into a topological space Y is an open mapping if $f(U)$ is an open subset of Y, for every open subset U of X.

Theorem 2.9. (Open Mapping Theorem) *Let $X = (X, +_1, \circ_1, || \,.\, ||_1, K)$ be a hyperBanach space and $Y = (Y, +_2, \circ_2, || \,.\, ||_2, K)$ be a hyperBanach space such that $k \circ_2 0 = \{0\}$, for every $k \in K$, $y \in k^{-1} \circ_2 x$, for $x \in k \circ_2 y$, $0 \neq k \in K$, and $x, y \in Y$, and $k \circ_2 \overline{A} \subseteq \overline{k \circ_2 A}$, for every $k \in K$, and $A \subseteq Y$. Then every bounded strong homomorphism from X onto Y is an open mapping.*

Proof. Let $T : X \longrightarrow Y$ be an onto bounded strong homomorphism. Suppose that the image under T of the unit ball U of X is open. Let V be an open subset of X. If $x \in V$, then $x +_1 r \circ_1 U \subseteq V$, for some $r > 0$, and so $T(V)$ includes the neighborhood $T(x) +_2 r \circ_2 T(U)$ of $T(x)$. Thus it is enough to show that $T(U)$ is an open set.

For each $y \in Y$, let $p(y) = \inf\{\, ||x||_1 \,:\, x \in X, T(x) = y\,\}$. For $y \in Y$ and $\alpha \in K$, we have if $w \in X$ such that $T(w) = y$, then

$$\{\inf\{||x||_1 \,:\, x \in X, T(x) = z\} \,:\, z \in \alpha \circ_2 T(w) = T(\alpha \circ_1 w)\} \subseteq ||\alpha \circ_1 w||_1.$$

Hence

$$\begin{aligned} \sup p(\alpha \circ_2 y) &= \sup\{\inf\{||x||_1 : x \in X, T(x) = z\} : z \in \alpha \circ_2 T(w) = T(\alpha \circ_1 w)\} \\ &\leqslant \sup ||\alpha \circ_1 w||_1 = |\alpha| ||w||_1, \end{aligned}$$

and therefore

$$\sup p(\alpha \circ_2 y) \leqslant |\alpha| \inf\{\, ||w||_1 \,:\, w \in X, T(w) = y\,\} = |\alpha| p(y).$$

Now, let $\sum_{n=1}^{\infty} y_n$ converges in Y. For an arbitrary $\epsilon > 0$, let $\{x_n\}$ be a sequence in X that $T(x_n) = y_n$ and $||x_n|| < p(y_n) + 2^{-n}\epsilon$, for every $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} ||x_n||_1 < \sum_{n=1}^{\infty} p(y_n) + \epsilon$, a finite number. Since X is a hyperBanach space, the absolutely convergent series $\sum_{n=1}^{\infty} x_n$ is convergent. Now,

$$T\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} T(x_n) = \sum_{n=1}^{\infty} y_n,$$

and so

$$p\left(\sum_{n=1}^{\infty} y_n\right) \leqslant \left\|\sum_{n=1}^{\infty} x_n\right\|_1 \leqslant \sum_{n=1}^{\infty} ||x_n||_1 < \sum_{n=1}^{\infty} p(y_n) + \epsilon.$$

So, p is countably subadditive.

The other properties of the prenorm can easily be checked.

Thus, p is a countably subadditive prenorm on Y, and by Theorem 2.6, it is continuous. Finally,

$$T(U) = \{y \,:\, y \in Y, T(x) = y, \text{for some}\, x \in U\,\} = \{\, y \,:\, y \in Y, p(y) < 1\,\},$$

so $T(U)$ is open and the proof is complete. ■

A strongly homomorphism T between two normed hypervector spaces is called and *isomorphism* if it is one-to-one and continuous and its inverse mapping T^{-1} is continuous on the range of T.

Corollary 2.10. *Let* $X = (X, +_1, \circ_1, ||\cdot||_1, K)$ *be a hyperBanach space and* $Y = (Y, +_2, \circ_2, ||\cdot||_2, K)$ *a hyperBanach space be such that* $k \circ_2 0 = \{0\}$, *for every* $k \in K$, $y \in k^{-1} \circ_2 x$, *for* $x \in k \circ_2 y$, $0 \neq k \in K$, *and* $x, y \in Y$, *and* $k \circ_2 \overline{A} \subseteq \overline{k \circ_2 A}$, *for every* $k \in K$, *and* $A \subseteq Y$. *Then every one-to-one bounded strongly homomorphism from* X *onto* Y *is an isomorphism.*

Theorem 2.11. (The Uniform Boundedness Principle Theorem) *Suppose that* $X = (X_1, +_1, \circ_1, ||\ .\ ||_1, K)$ *is a hyperBanach space such that* $k \circ_1 0 = \{0\}$, *for every* $k \in K$, $y \in k^{-1} \circ_1 x$, *for* $x \in k \circ_1 y$, $0 \neq k \in K$, *and* $x, y \in X$, *and* $k \circ_1 \overline{A} \subseteq \overline{k \circ_1 A}$, *for every* $k \in K$, *and* $A \subseteq X$, *and* $Y = (Y, +_2, \circ_2, ||\cdot||_2, K)$ *a normed hypervector space. Let* $\mathfrak{F}$ *be a non-empty family of bounded strong homomorphisms from* X *into* Y. *If* $\sup\{\,||T(x)||\ :\ T \in \mathfrak{F}\,\}$ *is finite for every* $x \in X$, *then* $\sup\{||T|| : T \in \mathfrak{F}\}$ *is finite.*

Proof. Let $p(x) = \sup\{||T(x)||_2 : T \in \mathfrak{F}\}$, for every $x \in X$. Then we have

$$\begin{aligned}
\sup p(\alpha \circ_1 x) &= \sup\{\sup\{||T(y)||_2 : T \in \mathfrak{F}\} :\ y \in \alpha \circ_1 x\}\\
&= \sup\{|\alpha|||T(x)||_2 : T \in \mathfrak{F},\} = |\alpha| \sup\{||T(x)||_2 : T \in \mathfrak{F}\}\\
&= |\alpha| p(x).
\end{aligned}$$

If $\sum_{n=1}^{\infty} x_n$ is a convergent series in X and $T \in \mathfrak{F}$, then

$$\left\|T\left(\sum_{n=1}^{\infty} x_n\right)\right\|_2 = \left\|\sum_{n=1}^{\infty} T(x_n)\right\|_2 \leqslant \sum_{n=1}^{\infty} ||T(x_n)||_2 \leqslant \sum_{n=1}^{\infty} p(x_n),$$

from which it follows that

$$p\left(\sum_{n=1}^{\infty} x_n\right) \leqslant \sum_{n=1}^{\infty} p(x_n).$$

So p is countably subadditive. Also we have

$$||T(x - Y)||_2 = \sup ||(-1) \circ_2 T(y - x)||_2 = ||T(y - x)||_2,$$

for every x, $y \in X$. So $p(x - y) = p(y - x)$. Since $p(0) = 0$, then p is a countably subadditive prenorm on X. Therefore, by Theorem 2.6, p is continuous and there is $\delta > 0$ such that $p(x) \leqslant 1$ whenever $||x||_1 < \delta$. It follows that $p(x) \leqslant \delta^{-1}$ whenever $||x||_1 < 1$, and therefore $||T(x)||_2 \leqslant \delta^{-1}$ whenever $T \in \mathfrak{F}$ and $||x||_1 < 1$, that is $||T||_2 \leqslant \delta^{-1}$, for each $T \in \mathfrak{F}$. The proof is complete. ∎

Theorem 2.12. (Closed Graph Theorem) *Let* $X = (X_1, +_1, \circ_1, ||\ .\ ||_1, K)$ *be a hyperBanach space such that* $k \circ_1 0 = \{0\}$, *for every* $k \in K$, $y \in k^{-1} \circ_1 x$, *for* $x \in k \circ_1 y$, $0 \neq k \in K$, *and* x, $y \in X$, *and* $k \circ_1 \overline{A} \subseteq \overline{k \circ_1 A}$, *for every* $k \in K$, *and* $A \subseteq X$, $(Y, +_2, \circ_2, ||\ .\ ||_2, K)$ *a hyperBanach space, and* T *a strong homomorphism* *from X into Y. Suppose that whenever a sequence $\{x_n\}$ in X converges to some x in X and $\{T(x_n)\}$ converges to some y in Y, it follows that $y = T(x)$. Then T is bounded.*

Proof. Let $p(x) = ||T(x)||_2$, for every $x \in X$. If we prove that p is continuous, then there is a neighborhood U of 0 such that the set $p(U)$ is bounded and therefore $T(U)$ is bounded. Let $r > 0$ be small enough that the closed ball of radius r and center 0, $C_r(0)$, is include in U, and let $M_0 = \sup\{\, ||T(x)||_2 \,:\, x \in C_r(0)\,\}$. If $0 \neq x \in X$, then

$$(r||x||_1^{-1}) \circ x \subseteq C_r(0),$$

and so

$$(r||x||_1^{-1})||T(x)||_2 = \sup ||T((r||x||_1^{-1}) \circ_1 x)|| \leqslant M_0,$$

and therefore $||T(x)||_2 \leqslant r^{-1}M_0||x||_1$. So T is bounded. Hence by Theorem 2.6, it is enough to show that p is a countably subadditive prenorm on X. Clearly, $\sup p(k \circ_1 x) = |k|p(x)$, $p(0) = 0$, and by the proof of the previous theorem, $p(x-y) = p(y-x)$. So, let $\sum_{n=1}^{\infty} x_n$ be a convergent series in X. Without loss of generality, we may assume that

$$\sum_{n=1}^{\infty} ||T(x_n)||_2 < \infty.$$

Since Y is a hyperBanach space, then $\sum_{n=1}^{\infty} T(x_n)$ converges. Also by the hypothesis,

$$\lim_{n\longrightarrow\infty} \sum_{n=1}^{m} x_n = \sum_{n=1}^{\infty} x_n,$$

and

$$\lim_{n\longrightarrow\infty} T\left(\sum_{n=1}^{m} x_n\right) = \sum_{n=1}^{m} T(x_n) = \sum_{n=1}^{\infty} T(x_n)$$

imply that

$$\sum_{n=1}^{\infty} T(x_n) = T\left(\sum_{n=1}^{\infty} x_n\right).$$

Therefore

$$\left\|T\left(\sum_{n=1}^{\infty} x_n\right)\right\|_2 = \left\|\sum_{n=1}^{\infty} T(x_n)\right\|_2 \leqslant \sum_{n=1}^{\infty} ||T(x_n)||_2,$$

which shows that p is countably subadditive and the proof is complete. ∎

3. Quasi norm and factor hypervector spaces

Definition 3.1. Let $X = (X, +, \circ, K)$ be a hypervector space. Suppose that for every $a \in K$, $|a|$ denoted the valuation of a in K. A quasinorm in X is a mapping

$$|| \cdot || : X \longrightarrow \mathbb{R}$$

that for all $a \in K$ and x, $y \in X$ has the following properties:

(i) $||x|| = 0 \Longleftrightarrow x = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$,

(iii) $\sup ||a \circ x|| \leqslant |a| \, ||x||$.

Definition 3.2. A quasinormed hypervector space $X = (X, +, \circ, || \,.\, ||, K)$ is called well if for every $x \in X$ and a, $b \in K$,

$$\inf ||a \circ x|| \leqslant \sup ||b \circ x||.$$

Example 3.3. It is easily seen that $(\mathbb{R}^n, +, \circ, || \cdot ||, \mathbb{R})$ that is defined in Example 2.2, is a quasinormed hypervector space that is well, since for every $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $\inf ||a \circ x|| = 0$.

Theorem 3.4. *Let $X = (X, +, \circ, || \cdot ||, K)$ be a well normed hypervector space and Y be a closed subspace of X. Then $(X/Y, +, \circ, || \cdot ||, K)$ is a quasinormed hypervector space, where for every $[x] \in X/Y$,*

$$||[x]|| = \inf\{||x + y|| : y \in Y\}.$$

Proof. Let $\lambda \in K$ and $[x]$, $[y] \in X/Y$. If $[x] = [0]$, then $x \in Y$ and therefore

$$0 \leqslant ||[x]|| \leqslant ||x - x|| = 0.$$

Conversely, $||[x]|| = 0$ implies that there is a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} ||x + x_n|| = 0$. Since Y is closed, it follows that x is in Y so $[x] = [0]$. Further, we have

$$\begin{aligned} ||[x] + [y]|| &= ||[x + y]|| = \inf\{\, ||x + y + z|| \,:\, z \in Y \,\} \\ &\leqslant \inf\{\, ||x + z_1 + y + z_2|| \,:\, z_1,\, z_2 \in Y \,\} \\ &\leqslant \inf\{\, ||x + z_1|| \,:\, z_1 \in Y \,\} + \inf\{\, ||y + z_2|| \,:\, z_2 \in Y \,\} \\ &= ||[x]|| + ||[y]||. \end{aligned}$$

At last,

$$\begin{aligned}\sup \|\lambda \circ [x]\| &= \sup\{ \|[z]\| \,:\, z \in \lambda \circ x \} \\ &= \sup\{\inf\{\|z + y\| \,:\, y \in Y \,:\, z \in \lambda \circ x \} \\ &\leqslant \sup\{\inf\{ \|\lambda \circ x + \lambda \circ y\| \,:\, y \in Y \} \\ &\leqslant \inf\{\sup\{ \|\lambda \circ x + \lambda \circ y\| \,:\, y \in Y \} \\ &\leqslant \inf\{\lambda\|x + y\| \,:\, y \in Y \} = \lambda\|[x]\|.\end{aligned}$$

Therefore $\| \,.\, \|$ is a quasinorm on X/Y and the proof is complete. ∎

Finally, the following lemmas can be proved easily similar as the normed vector spaces. So, we omit their proofs.

Theorem 3.5. *Let Y be a closed subspace of a well normed hypervector space $X = (X, +, \circ, \| \,.\, \|, K)$ and $F : X \longrightarrow X/Y$ the quotient map defined by $F(x) = x + Y$. Then F is continuous and maps open sets in X onto open sets in X/Y.*

Theorem 3.6. *Let Y be a closed subspace of a well normed hypervector space $X = (X, +, \circ, \| \,.\, \|, K)$. If X is a hyperBanach space, then so is X/Y.*

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