SOME RESULTS ON ANALOGOUS CONTINUED FRACTION OF RAMANUJAN

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Abstract. We give a differential for $\frac{1}{C(q)}$ and prove an identity which is analogous to Ramanujan's Entry 3.2.7. We also give a simpler proof for Entry 9(v).

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1. Introduction

The Rogers-Ramanujan continued fraction, defined by

$$R(q) = \frac{q^{\frac{1}{5}}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots, \quad |q|<1. \tag{1.1}$$

first appeared in a paper by L.J. Rogers [6] in 1894. In his first two letters to G.H. Hardy [5, pp xxvii, xxviii], Ramanujan communicated several theorems on $R(q)$. Ramanujan's major work on continued fraction centres around the continued fraction $R(q)$. I find the analogous continued fraction $C(q)$ defined by

$$C(q) = \frac{1}{1+}\frac{1+q}{1+}\frac{q^2}{1+}\frac{q+q^3}{1+}\frac{q^4}{1+}\cdots = \frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)^2_\infty} \tag{1.2}$$

equally interesting. In this paper, we have given some results for $C(q)$ notably a differential for the reciprocal for $C(q)$ which is analogous to Entry 9(v) [4, p.258] for $R(q)$. In [2, p. 259], Berndt has proved this Entry. I have given a simpler proof of this Entry. Further I have considered two sets of relations which are equivalent and have proved an identity which is analogous to Ramanujan's Entry 3.2.7 [1, p.89].

2. Notations

We shall be using the customary q-product notation. Thus

For $|q| < 1$

$$(a)_0 = (a;q)_0 = 1 \text{ and for } n \geq 1,$$
$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

Furthermore,

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad |q| < 1.$$

If the base q is understood we use $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

Ramanujan's general theta function $f(a,b)$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty\,(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1 \tag{2.1}$$

and

$$f(-q) = f(-q,-q^2) = (q;q)_\infty, \quad |q| < 1. \tag{2.2}$$

3. A result for the differential of $\dfrac{1}{C(q)}$

By definition,

$$C(q) = \frac{1}{1+}\,\frac{1+q}{1+}\,\frac{q^2}{1+}\,\frac{q+q^3}{1+}\,\frac{q^4}{1+\ldots.} = \frac{(q;q^4)_\infty(q^3;q^4)_\infty}{(q^2;q^4)^2_\infty} \tag{3.1}$$

Let $F(q) = q^{-\frac{1}{8}}C(q)$ and $A(q) = \dfrac{1}{F(q)}$. Then

$$8q\frac{d}{dq}\left(\log\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\right) = 1 - \frac{(q;q)^4_\infty}{(-q;q)^4_\infty}. \tag{3.2}$$

Proof. Now

$$A(q) = q^{\frac{1}{8}}\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}\,.$$

Taking logarithmic differentiation with respect to q, we get

$$\frac{8q}{A(q)}\frac{d}{dq}A(q) = 1+8\sum_{n=0}^{\infty}\left\{\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right\}. \tag{3.3}$$

Using the result Srivastava [7],

$$1-8\sum_{n=0}^{\infty}\left\{\frac{(4n+1)q^{4n+1}}{1-q^{4n+1}} - \frac{2(4n+2)q^{4n+2}}{1-q^{4n+2}} + \frac{(4n+3)q^{4n+3}}{1-q^{4n+3}}\right\} = \frac{(q;q)^4_\infty}{(-q;q)^4_\infty},$$

equation (3.3) becomes

$$\frac{8q}{A(q)}\frac{d}{dq}\left(q^{\frac{1}{8}}\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)=1+1-\frac{(q;q)_\infty^4}{(-q;q)_\infty^4}$$

or

$$\frac{8q}{A(q)}\left[\frac{1}{8q}+\frac{d}{dq}\left(\log\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)\right]A(q)=1+1-\frac{(q;q)_\infty^4}{(-q;q)_\infty^4}$$

or

$$8q\frac{d}{dq}\left(\log\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}\right)=1-\frac{(q;q)_\infty^4}{(-q;q)_\infty^4},$$

which proves (3.2).

Simpler proof of Entry 9(v) [4, p. 258]

Entry 9(v):

If

$$R(q)=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...},$$

then

$$1-\frac{f^5(-q)}{f(-q^5)}=5q\frac{d}{dq}\log\frac{f(-q^2,-q^3)}{f(-q,-q^4)}. \tag{3.4}$$

Proof. By [1, eq. (1.1.1), p. 9],

$$R(q)=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{f(-q,-q^4)}{f(-q^2,-q^3)}. \tag{3.5}$$

Let $G(q)=q^{-\frac{1}{5}}R(q)$ and $B(q)=\dfrac{1}{G(q)}$, then

$$B(q)=q^{\frac{1}{5}}\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Taking logarithmic differentiation of (3.5) with respect to q, we have

$$\begin{aligned}\frac{5q}{B(q)}\frac{d}{dq}[B(q)] &= 1+5\sum_{n=0}^{\infty}\left\{\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}\right.\\ &\left.-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right\}.\end{aligned} \tag{3.6}$$

Using the following formula of Ramanujan [2]

$$\begin{aligned}1-5\sum_{n=0}^{\infty}&\left\{\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}\right.\\ &\left.-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right\}=\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}=\frac{f^5(-q)}{f(-q^5)},\end{aligned} \tag{3.7}$$

equation (3.6) can be written as

$$\frac{5q}{B(q)}\frac{d}{dq}\left[B(q)\right]=1+1-\frac{f^5(-q)}{f(-q^5)} \tag{3.8}$$

or

$$\frac{5q}{B(q)}\left[\frac{1}{5q}+\frac{d}{dq}\log\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\right]B(q)=1+1-\frac{f^5(-q)}{f(-q^5)}$$

or

$$5q\frac{d}{dq}\log\frac{f(-q^2,-q^3)}{f(-q,-q^4)}=1-\frac{f^5(-q)}{f(-q^5)}$$

we have the Entry 9(v)[4, p.258].

4. A transformation formula for $C(q)$

Ramanujan in the "lost" note book has stated the formula

$$\frac{f(-\lambda^2q^3,-\lambda q^6)+qf(-\lambda,-\lambda^2q^9)}{f(-\lambda q^3)}=\frac{f(-q^2,-\lambda q)}{f(-q,-\lambda q^2)}.$$

This formula is extensively used by Ramanujan in his note book, but the statement is found only in the "lost" note book.

Take $\lambda=q$

$$\frac{f(-q^5,-q^7)+qf(-q,-q^{11})}{f(-q^4)}=\frac{f(-q^2,-q^2)}{f(-q,-q^3)}. \tag{4.1}$$

Using (2.1) and then (1.2), we get

$$\frac{f(-q^5,-q^7)+qf(-q,-q^{11})}{f(-q^4)}=\frac{(q^2;q^4)^2_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty}=\frac{1}{C(q)}. \tag{4.2}$$

5. Two equivalent relations

Let $u=C(q)$ and $v=C(q^4)$.

(i) $$2u=B+2^8q\frac{f^8(-q^4)}{f^8(-q)} \tag{5.1}$$

and

$$2u=B+\frac{f^8(-q)}{qf^8(-q^4)} \tag{5.2}$$

are equivalent.

(ii) $2v = K + 4q^{\frac{5}{8}}\dfrac{f(-q^{16})}{f(-q)}$ (5.3)

and

$$2v = K + \frac{f(-q^{\frac{1}{4}})}{q^{\frac{5}{32}}f(-q^4)} \tag{5.4}$$

are equivalent.

Proof of (i)

We shall use Ramanujan's transformation formula [2, p. 270, eq. 12.10]

$$e^{-\frac{a}{12}}a^{\frac{1}{4}}f(-e^{-2a}) = e^{-\frac{b}{12}}b^{\frac{1}{4}}f(-e^{-2b}) \tag{5.5}$$

twice, first by taking $a = \dfrac{\alpha}{2}, b = \dfrac{\beta}{2}$ and then by taking $a = 2\alpha,\ b = \dfrac{\beta}{8}$ to get

$$e^{-\frac{\alpha}{24}}\left(\frac{\alpha}{2}\right)^{\frac{1}{4}}f(-e^{-a}) = e^{-\frac{\beta}{24}}\left(\frac{\beta}{2}\right)^{\frac{1}{4}}f(-e^{-\beta}) \tag{5.6}$$

and

$$e^{-\frac{4\alpha}{24}}\left(\frac{4\alpha}{2}\right)^{\frac{1}{4}}f(-e^{-4a}) = e^{-\frac{\beta}{96}}\left(\frac{\beta}{8}\right)^{\frac{1}{4}}f(-e^{-\frac{\beta}{4}}). \tag{5.7}$$

By (5.6) and (5.7), we have

$$e^{\frac{\alpha}{8}}\frac{f(-e^{-\alpha})}{f(-e^{-4\alpha})} = 2e^{-\frac{\beta}{32}}\frac{f(-e^{-\beta})}{f(-e^{-\frac{\beta}{4}})}. \tag{5.8}$$

Putting $q = e^{-\alpha}$ and $Q = e^{-\frac{\beta}{4}}$ in (5.8), we have

$$\frac{f(-q)}{q^{\frac{1}{8}}f(-q^4)} = 2Q^{\frac{1}{8}}\frac{f(-Q^4)}{f(-Q)}. \tag{5.9}$$

or

$$\frac{f^8(-q)}{qf^8(-q^4)} = 2^8Q\frac{f^8(-Q^4)}{f^8(-Q)} \tag{5.10}$$

Writing Q for q in (5.1), we have (5.2). Thus (5.1) and (5.2) are equivalent.

Proof of (ii)

Applying transformation formula (5.5) twice, first with $a = \dfrac{\alpha}{8},\ b = 2\beta$ and then with $a = 2\alpha$, $b = \dfrac{\beta}{8}$ and then taking $q = e^{-\alpha}, Q^4 = e^{-\beta}$, we get

$$q^{-\frac{5}{32}}\frac{f(-q^{\frac{1}{4}})}{f(-q^4)} = 4Q^{\frac{5}{8}}\frac{f(-Q^{16})}{f(-Q)}. \tag{5.11}$$

Replacing q by Q in (5.3), we have (5.4). Thus (5.3) and (5.4) are equivalent.

6. Another identity

We prove the following identity which is analogous to Ramanujan's Entry 3.2.7 (p 364) [1, p 89].

If

$$q^{-1}C^8(q) - \frac{2}{q^{-1}C^8(q)} = -2\mu, \tag{6.1}$$

then

$$q^{-\frac{1}{8}}C(q) = \left[\left(\mu^2+2\right)^{\frac{1}{2}} - \mu\right]^{\frac{1}{8}}. \tag{6.2}$$

Proof. Let

$$J = q^{-1}C^8(q),$$

then (6.1) is

$$J - \frac{2}{J} = -2\mu, \quad \text{or} \quad J^2 + 2\mu J - 2 = 0.$$

Solving for J

$$J = \left(\mu^2+2\right)^{\frac{1}{2}} - \mu$$

$$\left[q^{-\frac{1}{8}}C(q)\right]^8 = \left(\mu^2+2\right)^{\frac{1}{2}} - \mu$$

$$q^{-\frac{1}{8}}C(q) = \left[\left(\mu^2+2\right)^{\frac{1}{2}} - \mu\right]^{\frac{1}{8}}.$$

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