SOME DIVISIBLE MATRIX GROUPS

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Abstract. A group G is said to be *divisible* if for any $x \in G$ and any positive integer n, there exists an element $y \in G$ such that $x = y^n$. For a positive integer $n > 1$, let $G_n(\mathbb{R})$ be the group under multiplication of all invertible $n \times n$ matrices over $\mathbb{R}$. For distinct positive integers $p, q \leq n$, let $G(n, p, q)$ be the subgroup of $G_n(\mathbb{R})$ consisting of all $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \ldots, n\}$ and $A_{ij} = 0$ for distinct i, j such that $(i, j) \neq (p, q)$. Also, let $U(n)[L(n)]$ be the subgroup of $G_n(\mathbb{R})$ consisting of all upper [lower] triangular matrices $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \ldots, n\}$. The purpose of this paper is to show that the matrix groups $G(n, p, q), U(n)$ and $L(n)$ are all divisible.

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ be respectively the set of all positive integers and the set of all real numbers. The set of all positive real numbers is denoted by $\mathbb{R}^+$.

A semigroup S is called *divisible* if for every $x \in S$ and $n \in \mathbb{N}$, $x = y^n$ for some $y \in S$. Note that the commutative groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are divisible but $(\mathbb{R} \backslash \{0\}, \cdot)$ is not divisible. Certain divisible semigroups have long been studied. For example, see [6], [1], [2], [9] and [8]. In [8], the authors gave characterizations determining when some periodic semigroups are divisible. Divisible commutative groups are characterized in terms of injectivity. This can be seen in [4], pp.195-196.

For $n \in \mathbb{N}$, let $M_n(\mathbb{R})$ be the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{R}$. The entry of $A \in M_n(\mathbb{R})$ in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column will be denoted by A_{ij}. Let $G_n(\mathbb{R})$ be the unit group of $M_n(\mathbb{R})$, that is, $G_n(\mathbb{R})$ is the multiplicative group of all invertible $n \times n$ matrices over $\mathbb{R}$, or equivalently, $G_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$. Clearly, $M_n(\mathbb{R})$ is not a divisible semigroup and $G_n(\mathbb{R})$ is not a divisible group. In [7] and [5], the authors introduced a skew-semifield SK_n which is neither a semifield nor a skew-field as follows : Let $n \in \mathbb{N}$ and $n > 1$. If SK_n is the set of all $n \times n$ matrices of the form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & b \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

where $a_1, \dots, a_n \in \mathbb{R}^+$ and $b \in \mathbb{R}$, then SK_n with the $n \times n$ zero matrix under the usual addition $+$ and multiplication $\cdot$ of matrices is a skew-semifield which is neither a skew-field nor a semifield. Recall that a semiring $(S, +, \cdot)$ is called a *skew-semifield* if $(S, +)$ is a commutative semigroup with identity 0 and $(S, \cdot)$ is a group with zero 0. A *semifield* is a commutative skew-semifield. Note that SK_n is a noncommutative subgroup of the group $G_n(\mathbb{R})$ and

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & b \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \dots & \frac{-b}{a_1 a_n} \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{a_n} \end{bmatrix}$$

Since $(\mathbb{R}^+, \cdot)$ is a divisible group and every entry of any $A \in SK_n$ in the diagonal is positive, it is natural to ask whether the group SK_n is divisible. We first generalize the group SK_n as follows : For distinct positive integers $p, q \leq n$, let $G(n, p, q)$ be the set of all $A \in M_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \dots, n\}$ and $A_{ij} = 0$ for distinct $i, j \in \{1, \dots, n\}$ such that $(i, j) \neq (p, q)$. Then $G(n, 1, n) = SK_n$. It is straightforward to check that $G(n, p, q)$ is a noncommutative semigroup. If $A \in G(n, p, q)$, then $B \in G(n, p, q)$ defined

$$B_{ij} = \begin{cases} \dfrac{1}{A_{ij}} & \text{if } i = j, \\ -\dfrac{A_{pq}}{A_{pp}A_{qq}} & \text{if } i = p \text{ and } j = q, \\ 0 & \text{otherwise} \end{cases}$$

is an inverse of A in $G_n(\mathbb{R})$. Therefore $G(n, p, q)$ is a noncommutative subgroup of $G_n(\mathbb{R})$. The first main interest in this paper is to show that the group $G(n, p, q)$ is always divisible.

It can be seen from [3], page 410 that the set $\{A \in G_n(\mathbb{R}) \mid A \text{ is upper [lower] triangular}\}$ forms a subgroup of $G_n(\mathbb{R})$. This fact and the divisibility of the group $G(n, p, q)$ motivate us to consider the set $U(n)[L(n)]$ of all upper [lower] triangular matrices $A \in G_n(\mathbb{R})$ with $A_{ii} > 0$ for all $i \in \{1, \dots, n\}$. Note that for $A, B \in U(n)[L(n)]$, $\det A = A_{11}A_{22} \dots A_{nn} > 0$, $(AB)_{ii} = A_{ii}B_{ii} > 0$ and $(A^{-1})_{ii} = \frac{1}{A_{ii}} > 0$ for all $i \in \{1, \dots, n\}$. Therefore we deduce that both $U(n)$ and $L(n)$ are subgroups of $G_n(\mathbb{R})$. Also, $U(n)$ and $L(n)$ are noncommutative.

Moreover, $G(n,p,q) \subseteq U(n)$ if $p < q \leq n$ and $G(n,p,q) \subseteq L(n)$ if $q < p \leq n$. Our second purpose is to show that the groups $U(n)$ and $L(n)$ are also divisible.

In the remainder of this paper, let $n \in \mathbb{N}$ with $n > 1$, $G(n,p,q)$ where $p \neq q$ and $p, q \leq n$, $U(n)$ and $L(n)$ be the matrix groups over $\mathbb{R}$ mentioned above. Recall that if A is an element of $G(n,p,q)$, $U(n)$ or $L(n)$, then

$$(A^m)_{ii} = A_{ii}^m \text{ for all } m \in \mathbb{N} \text{ and } i \in \{1, \ldots, n\}.$$

2. The matrix group $G(n,p,q)$

To show that $G(n,p,q)$ is a divisible group, the following lemma is a main tool.

Lemma 2.1. *If $A \in G(n,p,q)$ and $m \in \mathbb{N}$, then*

$$(A^m)_{pq} = \Big(\sum_{i=0}^{m-1} A_{pp}^i A_{qq}^{m-1-i}\Big) A_{pq}.$$

Proof. First, assume that $p < q$. Since $A_{ij} = 0$ for all $i, j \in \{1, ..., n\}$ with $i > j$, we have

$$(A^m)_{pq} = \sum_{p=t_1 \leq t_2 \leq \ldots \leq t_{m+1}=q} A_{t_1t_2}A_{t_2t_3}...A_{t_mt_{m+1}}.$$

By the property of A, $A_{pi} = A_{iq} = 0$ for all $i \in \{1, ...s, n\}$ with $p < i < q$. It follows that

$$\begin{aligned}(A^m)_{pq} &= \sum_{\substack{p=t_1 \leq t_2 \leq \ldots \leq t_{m+1}=q \\ t_r \in \{p,q\} \text{ for all } r}} A_{t_1t_2}A_{t_2t_3} \ldots A_{t_mt_{m+1}} \\ &= \sum_{k=0}^{m-1} (A_{pp}^k A_{pq} A_{qq}^{m-1-k}) = \Big(\sum_{k=0}^{m-1} A_{pp}^k A_{qq}^{m-1-k}\Big) A_{pq}.\end{aligned}$$

It can be proved analogously for the case that $p > q$. ■

Theorem 2.2. *The group $G(n,p,q)$ is divisible.*

Proof. Let $A \in G(n,p,q)$ and $m \in \mathbb{N}$. Recall that $A_{ii} > 0$ for every $i \in \{1, ..., n\}$. Define $B \in G(n,p,q)$ by

$$\begin{aligned}B_{ii} &= A_{ii}^{\frac{1}{m}} \text{ for all } i \in \{1, \ldots, n\}, \\ B_{pq} &= \frac{A_{pq}}{\displaystyle\sum_{k=0}^{m-1} B_{pp}^k B_{qq}^{m-1-k}}\end{aligned}$$

Then for every $i \in \{1, \ldots, n\}$, $(B^m)_{ii} = B_{ii}^m = (A_{ii}^{\frac{1}{m}})^m = A_{ii}$. From Lemma 2.1 and the definition of B_{pq}, we have

$$(B^m)_{pq} = \Big(\sum_{k=0}^{m-1} B_{pp}^k B_{qq}^{m-1-k}\Big) B_{pq} = A_{pq}.$$

This shows that $A = B^m$, as desired. ∎

3. The matrix groups $U(n)$ and $L(n)$

The following lemma is needed to show that the matrix group $U(n)$ is divisible.

Lemma 3.1. *Let $A \in U(n)$ and $m \in \mathbb{N}$. Then the following statements hold.*

(i) *For $i \in \{1, ..., n-1\}$,*

$$(A^m)_{i,i+1} = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+1,i+1}^{m-1-k}\Big) A_{i,i+1}.$$

(ii) *If $n > 2$ and $2 \le l < n$, then, for $i \in \{1, \ldots, n-l\}$,*

$$(A^m)_{i,i+l} = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+l,i+l}^{m-1-k}\Big) A_{i,i+l} + \sum_{\substack{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}}.$$

Proof. (i) Since A is upper triangular, we have that for $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}(A^m)_{i,i+1} &= \sum_{i=t_1\le t_2\le\ldots\le t_{m+1}=i+1} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}} \\ &= \sum_{k=0}^{m-1}\big(A_{ii}^k A_{i,i+1} A_{i+1,i+1}^{m-1-k}\big) = \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+1,i+1}^{m-1-k}\Big) A_{i,i+1}\end{aligned}$$

(ii) Assume that $n > 2$ and $2 \le l < n$. Then for $i \in \{1, \ldots, n-l\}$,

$$\begin{aligned}(A^m)_{i,i+l} &= \sum_{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}} \\ &= \sum_{\substack{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l \\ t_{r+1}-t_r=l \text{ for some } r}} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}} \\ &\quad + \sum_{\substack{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}} \\ &= \sum_{k=0}^{m-1} A_{ii}^k A_{i,i+l} A_{i+l,i+l}^{m-1-k} + \sum_{\substack{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}} \\ &= \Big(\sum_{k=0}^{m-1} A_{ii}^k A_{i+l,i+l}^{m-1-k}\Big) A_{i,i+l} + \sum_{\substack{i=t_1\le t_2\le\ldots\le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} A_{t_1t_2}A_{t_2t_3}\ldots A_{t_mt_{m+1}}.\end{aligned}$$ ∎

Theorem 3.2. *The matrix group $U(n)$ is divisible.*

Proof. Let $A \in U(n)$ and $m \in \mathbb{N}$. Define $B \in U(n)$ by defining B_{ij} with $i \le j$ recursively on entries of the lines parallel to the diagonal as follows : Let

$$B_{ii} = A_{ii}^{\frac{1}{m}} \quad \text{for all } i \in \{1, \ldots, n\}$$

and let

$$B_{i,i+1} = \frac{A_{i,i+1}}{\sum_{k=0}^{m-1} B_{ii}^k B_{i+1,i+1}^{m-1-k}} \quad \text{for all } i \in \{1, \ldots, n-1\}.$$

If $i \in \{1, \ldots, n-2\}$, let

$$B_{i,i+2} = \frac{1}{\sum_{k=0}^{m-1} B_{ii}^k B_{i+2,i+2}^{m-1-k}} \Big(A_{i,i+2} - \sum_{\substack{i=t_1 \le t_2 \le \ldots \le t_{m+1}=i+2 \\ t_{r+1}-t_r<2 \text{ for all } r}} B_{t_1 t_2} B_{t_2 t_3} \ldots B_{t_m t_{m+1}} \Big).$$

Let $2 < l < n$ and assume that B_{ij} is defined for all $i, j \in \{1, \ldots, n\}$ with $i \le j$ and $j - i < l$. If $i \in \{1, \ldots, n-l\}$, define $B_{i,i+l}$ by

$$B_{i,i+l} = \frac{1}{\sum_{k=0}^{m-1} B_{ii}^k B_{i+l,i+l}^{m-1-k}} \Big(A_{i,i+l} - \sum_{\substack{i=t_1 \le t_2 \le \ldots \le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} B_{t_1 t_2} B_{t_2 t_3} \ldots B_{t_m t_{m+1}} \Big).$$

We claim that $B^m = A$. If $i \in \{1, \ldots, n\}$, then $(B^m)_{ii} = B_{ii}^m = (A_{ii}^{\frac{1}{m}})^m = A_{ii}$. Let $i, j \in \{1, \ldots, n\}$ be such that $i < j$ and let $l = j - i$.

Case 1 : $l = 1$. It follows from Lemma 3.1(i) that

$$(B^m)_{ij} = (B^m)_{i,i+1} = \Big(\sum_{k=0}^{m-1} B_{ii}^k B_{i+1,i+1}^{m-1-k} \Big) B_{i,i+1},$$

so by the definition of $B_{i,i+1}$, $(B^m)_{ij} = (B^m)_{i,i+1} = A_{i,i+1} = A_{ij}$.

Case 2 : $l > 1$. By Lemma 3.1(ii) and the definition of $B_{i,i+l}$, we have

$$\begin{aligned} (B^m)_{ij} = (B^m)_{i,i+l} = {} & \Big(\sum_{k=0}^{m-1} B_{ii}^k B_{i+l,i+l}^{m-1-k} \Big) B_{i,i+l} \\ & + \sum_{\substack{i=t_1 \le t_2 \le \ldots \le t_{m+1}=i+l \\ t_{r+1}-t_r<l \text{ for all } r}} B_{t_1 t_2} B_{t_2 t_3} \ldots B_{t_m t_{m+1}}. \end{aligned}$$

The last equality and the definition of $B_{i,i+l}$ yield the equality $(B^m)_{i,i+l} = A_{i,i+l}$. Thus $(B^m)_{ij} = A_{ij}$.

Therefore $B^m = A$ and hence the proof of the theorem is complete. ∎

For a matrix A, let A^t be the transpose of A. Then

$$L(n) = \{A^t \mid A \in U(n)\},$$

and hence the following result is obtained obviously from Theorem 3.2 and the facts that $(A^t)^t = A$ and $(A^t)^m = (A^m)^t$ for all $A \in M_n(F)$ and $m \in \mathbb{N}$.

Corollary 3.3. *The matrix group $L(n)$ is divisible.*

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