A NOTE ON QUASI $k$-IDEALS AND BI $k$-IDEALS IN TERNARY SEMIRINGS

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Abstract. Dutta and Kar [3] have introduced the concept of ternary semiring. In this paper, the notion of quasi $k$–ideals and bi $k$–ideals of a ternary semiring is introduced and characterizations $k$–regular ternary semirings has been given.

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1. Introduction and preliminaries

Ternary semiring is a generalization of a ternary ring which is introduced by Lister [6]. T.K. Dutta and S. Kar have initiated the notion of ternary semiring and studied their properties. Recall ([3], [4]) the followings. A non-empty set $S$ together with binary operation, called addition and ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $S$ is an additive commutative semigroup satisfying the following conditions:

(i) $abc \in S$,

(ii) $(abc)de = a(bcd)e = ab(cde)$,

(iii) $(a + b)cd = acd + bcd$,

(iv) $a(b + c)d = abd + acd$,

(v) $ab(c + d) = abc + abd$

for all $a, b, c, d, e \in S$.

Let $S$ be a ternary semiring. An element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$, then '0'is called the zero element.
of the ternary semiring $S$. In this case, $S$ is called ternary semiring $S$ with zero. Through out this paper, $S$ will always denote a ternary semiring with zero. An additive subsemigroup $I$ of $S$ is called left (right, lateral) ideal of $S$ if $s_1s_2a$ (respectively $as_1s_2, s_1as_2$) $\in I$ for all $s_1, s_2 \in S$ and $a \in I$. If $I$ is a left, a right and a lateral ideal of $S$ then $I$ is called an ideal of $S$. An ideal $I$ of $S$ is called a $k$–ideal if $x + y \in I, x \in S, y \in I$ imply that $x \in I$. If $A$ is an ideal of a ternary semiring $S$ then $\overline{A}=\{a \in S : a + x \in A \text{ for some } x \in A\}$ is called $k$–closure of A. It can easily verified that an ideal $A$ of $S$ is $k$–ideal if and only if $A=\overline{A}$ and also $A \subseteq \overline{A}, A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. Let $A, B, C$ be three subsets of $S$. Then $ABC$ denotes the set of all finite sums of the forms $\sum a_ib_ic_i$, where $a_i \in A, b_i \in B, c_i \in C$. An element $a \in S$ is called regular if there exists $x \in S$ such that $a = axa$. If all the elements of $S$ are regular then $S$ is called regular ternary semiring. An additive subsemigroup $Q$ of a ternary semiring $S$ is called a quasi-ideal of $S$ if $QSS \cap (SQS + SSQSS) \subseteq SSQ \subseteq Q$. Clearly, every quasi ideal of ternary semiring $S$ is a ternary subsemiring of $S$.

2. Main results

**Proposition 2.1.** Let $S$ be a ternary semiring and $X$ be a nonempty subset of $S$. Then

(i) $\langle X \rangle_l = Z_0^+X + SSX$ is the smallest left ideal generated by $X$,

(ii) $\langle X \rangle_r = Z_0^+X + XSS$ is the smallest right ideal generated by $X$,

(iii) $\langle X \rangle_t = Z_0^+X + SSX + XSS + SSXSS$ is the smallest two sided ideal generated by $X$,

(iv) $\langle X \rangle_m = Z_0^+X + SXS + SSXSS$ is the smallest lateral ideal generated by $X$,

(v) $\langle X \rangle_i = Z_0^+X + SSX + XSS + SXS + SSXSS$ is the smallest ideal generated by $X$,

where $SSX, XSS, SXS, SSXSS, Z_0^+X$ the set of all finite sum of the form $\sum r_is_ix_i, \sum x_ip_iq_i, \sum u_ix_iv_i, \sum p_iq_ix_ir_is_i, \sum n_ix_i$, where $r_i, s_i, p_i, q_i, u_i, v_i, p_i', q_i', r_i', s_i' \in S$, $x_i \in X, n_i \in Z_0^+$, and $Z_0^+$ is the set of all positive integer with zero.

The following corollary can be easily proved by the above proposition.

**Corollary 2.2.** If $X$ is a subsemigroup of $(S, +)$, then

$\langle X \rangle_l = X + SSX$,

$\langle X \rangle_r = X + XSS$,

$\langle X \rangle_t = X + SSX + XSS + SSXSS$,

$\langle X \rangle_m = X + SXS + SSXSS$,

$\langle X \rangle_i = X + SSX + XSS + SXS + SSXSS$. 
Proposition 2.3. [4] Let $S$ be a ternary semiring and $a \in S$. Then the principal

(i) left ideal generated by $a$ is given by

$$\langle a \rangle_l = \left\{ \sum r_i s_i a + na : r_i, s_i \in S : n \in Z_0^+ \right\}$$

(ii) right ideal generated by $a$ is given by

$$\langle a \rangle_r = \left\{ \sum ar_i s_i + na : r_i, s_i \in S : n \in Z_0^+ \right\}$$

(iii) two sided ideal generated by $a$ is given by

$$\langle a \rangle_t = \left\{ \sum r_i s_i a + \sum p_k' q_k' r_k' s_k' + na : r_i, s_i, p_i, q_i, r_k', s_k' \in S : n \in Z_0^+ \right\}$$

(iv) lateral ideal generated by $a$ is given by

$$\langle a \rangle_m = \left\{ \sum r_i a s_i + \sum p_j q_j r_j s_j + na : r_i, s_i, p_j, q_j, r_j, s_j \in S : n \in Z_0^+ \right\}$$

(v) ideal generated by $a$ is given by

$$\langle a \rangle_i = \left\{ \sum r_i s_i a + \sum p_k q_k r_k s_k + na : r_i, s_i, p_k, q_k, r_k, s_k \in S : n \in Z_0^+ \right\}$$

where $\sum$ denote the finite sum and $Z_0^+$ is the set of all positive integer with zero.

Proposition 2.4. If $Q$ is a quasi-ideal of a ternary semiring $S$, then

$$Q = Q + (SSQ \cap (SQS + SSQSS) \cap QSS).$$

Proof. The proof is obvious.

Let $X$ be a non empty subset of a ternary semiring $S$. The smallest quasi-ideal containing $X$ and generated by $X$ is denoted by $(X)_q$, that is, the intersection of all quasi-ideal of $S$ containing $X$.

Proposition 2.5. Let $S$ be a ternary semiring and $X$ be nonempty subset of $S$. Then $(X)_q = Z_0^+ X + (SSX \cap (SXS + SSXSS) \cap XSS)$.

Proof. Let $Q = Z_0^+ X + (SSX \cap (SXS + SSXSS) \cap XSS)$. Clearly, $Q$ is a non empty additive subsemigroup of $S$. Now,

$$(SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq SSQ$$

$$= SS(Z_0^+ X + (SSX \cap (SXS + SSXSS) \cap XSS))$$

$$\subseteq Z_0^+ SSX + SS(SSX) \subseteq SSX.$$
Similarly, \((SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq (SX + SSXSS)\) and \((SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq XSS\). Therefore

\[
(SSQ \cap (SQS + SSQSS) \cap QSS) \subseteq (SSX \cap (SX + SSXSS) \cap XSS) \subseteq Q.
\]

Hence \(Q\) is a quasi-ideal of \(S\) containing \(X\). Also, it is easy to show that \(Q\) is smallest quasi-ideal of \(S\) containing \(X\). Hence

\[
Q = (X)_q = Z_0^+X + (SSX \cap (SX + SSXSS) \cap XSS).
\]

**Theorem 2.6.** If \(Q\) is a quasi-ideal of a ternary semiring \(S\) and if \(Q \subseteq QSS\) and \(Q \subseteq SSQ\) and \(QSS\), \(SSQ\) are \(k\)-ideals of \(S\), then \(Q\) is the intersection of the left ideal \(Q + SSQ\), lateral ideal \(Q + SQS + SSQSS\), and right ideal \(Q + QSS\).

**Proof.** Let \(D = (Q + QSS) \cap (Q + SQS + SSQSS) \cap (Q + QSS)\). Clearly, \(Q \subseteq D\).

To show \(D \subseteq Q\). Now, \(Q \subseteq QSS\) and \(Q \subseteq SSQ\). Therefore

\[
D = QSS \cap (Q + SQS + SSQSS) \cap SSQ.
\]

Let \(d \in D\). Then \(d \in QSS\), \(d \in SSQ\) and

\[
d = q + \Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i
\]

for \(s'_i, s_i, p'_i, p''_i, p'''_i \in S\) and \(q'_i, q''_i \in Q\).

Since \(q \in Q \subseteq QSS\) and \(QSS\) is a \(k\)-ideal of \(S\), therefore

\[
\Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i \in QSS.
\]

Similarly,

\[
\Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i \in SSQ.
\]

Therefore,

\[
\Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i \in QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q.
\]

So,

\[
\Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i \in Q,
\]

which implies that \(d = q + \Sigma s'_i q'_i s_i + \Sigma p'_i q''_i q'''_i p'''_i \in Q\). Thus \(D \subseteq Q\). Hence \(D = Q\). ■

**Definition 2.7.** An additive subsemigroup \(Q\) of a ternary semiring \(S\) is called a quasi \(k\)-ideal of \(S\) if \(QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q\). Clearly, every quasi \(k\)-ideal is a quasi-ideal of \(S\).

It is easy to see that if \(R\) be right \(k\)-ideal, \(M\) be lateral \(k\)-ideal and \(L\) be left \(k\)-ideal of \(S\), then \(Q = R \cap M \cap L\) is a quasi \(k\)-ideal of \(S\), because \((R \cap M \cap L)SSS \cap SSSS(R \cap M \cap L)SSS \cap SS(R \cap M \cap L)SSS \cap SSL = R \cap M \cap L = R \cap M \cap L\).
Lemma 2.8. Let $S$ be a ternary semiring and $A, B, C \subseteq S$, then

$$ABC = \overline{AB}C.$$  

Proof. Since $A \subseteq \overline{A}$, $B \subseteq \overline{B}$ and $C \subseteq \overline{C}$, therefore $ABC \subseteq \overline{AB}C$. Hence, $\overline{ABC} \subseteq \overline{AB}C$. Again, let $x \in \overline{A}$, $y \in \overline{B}$ and $z \in \overline{C}$. Then there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $c_1, c_2 \in C$ such that $x + a_1 = a_2$, $y + b_1 = b_2$ and $z + c_1 = c_2$. Now,

\[
xyz + a_2 b_2 c_1 + a_2 b_1 c_2 + a_1 b_2 c_2 + a_1 b_1 c_1 \\
= xyz + (x + a_1)(y + b_1)c_1 + a_2 b_1 c_2 + a_1 b_2 c_2 + a_1 b_1 c_1 \\
= xyz + xyc_1 + xb_1 c_1 + a_1 yc_1 + a_1 b_1 c_1 + a_2 b_1 c_2 \\
+ a_1 b_2 c_2 + a_1 b_1 c_1 \\
= xyc_1 + xb_1 c_1 + a_1 yc_1 + a_1 b_1 c_1 + (x + a_1)b_1 c_2 \\
+ a_1 b_2 c_2 + a_1 b_1 c_1 \\
= x(y + b_1)c_2 + xb_1 c_1 + a_1(y + b_1)c_1 + a_1 b_1 c_2 \\
+ a_1 b_2 c_2 + a_1 b_1 c_1 \\
= xb_2 c_2 + (x + a_1)b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2 + a_1 b_2 c_2 \\
= (x + a_1)b_2 c_2 + a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2 \\
= a_2 b_2 c_2 + a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2.
\]

As $a_i b_j c_k(i,j,k = 1,2) \in ABC$, therefore we can prove that $xyz \in \overline{ABC}$, for $x \in \overline{A}, y \in \overline{B}, z \in \overline{C}$. Suppose $t = \sum_{finite} a_i b_j c_k$ for some $a_i \in \overline{A}$, $b_j \in \overline{B}$ and $c_k \in \overline{C}$. Thus $t \in \overline{ABC}$, i.e. $\overline{AB} \subseteq \overline{ABC}$. Therefore $\overline{ABC} \subseteq \overline{AB} = \overline{ABC}$. Hence $\overline{ABC} = \overline{ABC}$. 

Definition 2.9. [4] Let $S$ be a ternary semiring. Then $S$ is called $k$–regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Theorem 2.10. If a ternary semiring $S$ is $k$–regular, then every quasi $k$–ideal $Q$ of $S$ is of the form $Q = \overline{QSQS} = \overline{QSS} \cap \overline{SQS} + \overline{SSQS} \cap \overline{SSQ}$. 

Proof. Let $Q$ be a quasi $k$–ideal of $S$. Then $\overline{QSS} \cap \overline{SQS} + \overline{SSQS} \cap \overline{SSQ} \subseteq Q$. Let $a \in Q$ and $S$ is $k$–regular, then there exist $x, y \in S$ such that $a + axa = aya$. This implies that $axa + axaxa = ayaxa$. Since $axaxa, ayaxa \in \overline{QSSQ}$, therefore $axa \in \overline{QSSQ}$. Similarly, $aya \in \overline{QSSQ}$. Therefore $a \in \overline{QSSQ} = \overline{QSSQ}$ (as $\overline{QSSQ}$ is $k$–closed). Therefore $Q \subseteq \overline{QSSQ}$. Again $\overline{QSSQ} \subseteq Q(SSS)S \subseteq QSS$ and $\overline{QSSQ} \subseteq \overline{SSQ}$, therefore $Q \subseteq \overline{QSSQ} \subseteq \overline{QSSQ} \subseteq \overline{SSQS} \cap \overline{SSQ} \subseteq \overline{SSQS} \subseteq \overline{QSS}$, which shows that $\overline{QSSQ} \subseteq \overline{QSS}$, $\overline{QSSQ} \subseteq \overline{SSQ}$ and $\overline{QSSQ} \subseteq \overline{SSQS} \cap \overline{SSQ}$. Thus we have $Q \subseteq \overline{QSSQ} \subseteq \overline{QSS} \cap \overline{QSSQ} \subseteq \overline{QSSQ} \cap \overline{SSQS} \subseteq \overline{Q}$ (as $Q$ is a quasi $k$–ideal of $S$). Hence $\overline{QSSQ} = \overline{QSS} \cap \overline{QSSQ} \cap \overline{SSQS} \cap \overline{SSQ}$. 

Theorem 2.11. A ternary semiring \( S \) is \( k \)-regular if and only if \( R \cap M \cap L = RML \) holds for each right \( k \)-ideal \( R \), lateral \( k \)-ideal \( M \), and left \( k \)-ideal \( L \) of \( S \).

Proof. Since \( R \) is right \( k \)-ideal, therefore \( RML \subseteq RSS \subseteq R \) which shows that \( RML \subseteq R = R \). Again, since \( M \) is a lateral \( k \)-ideal of \( S \), then \( RML \subseteq M \subseteq M \) and so \( RML \subseteq M = M \). Similarly, we obtain that \( RML \subseteq L \). Therefore we have \( RML \subseteq R \cap M \cap L \).

Again, suppose that \( a \in R \cap M \cap L \). Since \( S \) is \( k \)-regular therefore there exist \( x, y \in S \) such that \( a + axa = aya \). This implies that \( axa + a(xax)a = a(yax)a \).
Since \( a(xax)a, a(yax)a \in RML \) therefore \( axa \in RML \). Similarly, \( aya \in RML \). Therefore \( a \in RML = RML \) (as \( RMS \) is \( k \)-closed). Hence \( R \cap M \cap L \subseteq RML \). Thus \( R \cap M \cap L = RML \).

Conversely, let \( a \in S \). Then principal right \( k \)-ideal generated by \( a \) of \( S \) is given by \( aSS + Z^+_0a \). Now,

\[
\begin{align*}
    aSS + Z^+_0a &= aSS + Z^+_0a \cap S \cap S \\
    &= aSS + Z^+_0aSS \text{ (as } S \text{ is itself (lateral, left) } k \text{-ideal of } S) \\
    &= (aSS + Z^+_0a)SS \text{ (by Lemma 2.8)} \\
    &= aSS.
\end{align*}
\]

Also, \( a = a0S + 1.a \in aSS + Z^+_0a \subseteq aSS + Z^+_0a = aSS \). Therefore \( a \in aSS \). Similarly, it can easily be shown that \( a \in SSa \) and \( a \in (SaS + SSaSS) \).

Therefore, we have

\[
\begin{align*}
    a \in aSS \cap (SaS + SSaSS) \cap SSa &= aSS (SaS + SSaSS) SSa \\
    &= aSS (SaS + SSaSS) SSa = aSa.
\end{align*}
\]

Therefore there exist \( x, y \in S \) such that \( a + axa = aya \). Thus \( S \) is \( k \)-regular. \( \blacksquare \)

Definition 2.12. A ternary subsemiring \( B \) of a ternary semiring \( S \) is called a bi \( k \)-ideal of of \( S \) if \( BSBSB \subseteq B \).

Result 2.13. Every quasi \( k \)-ideal of a ternary semiring is a bi \( k \)-ideal of \( S \).

Proof. It is obvious by Theorem 2.10. \( \blacksquare \)

Result 2.14. Let \( S \) be a \( k \)-regular ternary semiring. Then every bi \( k \)-ideal \( B \) of a ternary semiring is a quasi \( k \)-ideal of \( S \) if \( \overline{B} = B \) and \( BSB \subseteq B \).
Proof. Let $B$ be a bi $k$–ideal of $S$. Then

\[
BSS \cap (SBS + SSBSS) \cap SSB
= BSS(SBS + SSBSS)SSB \quad \text{(by Theorem 2.11)} \\
= BSS(SBS + SSBSS)SSB \quad \text{(by Lemma 2.8)} \\
= B(SSI)B(SSI)B + B(SSI)SBS(SSI)SB \\
\subseteq BSBSB + BSSBSSB \\
\subseteq BSBSB + B(SSI)SSB \\
\subseteq B + BSSB \quad \text{(as $B$ is a bi $k$–ideal of $S$)} \\
\subseteq B + B \quad \text{(by hypothesis).} \\
\subseteq B = B \quad \text{(by hypothesis).}
\]

Hence $B$ is a quasi $k$–ideal of $S$. □

Theorem 2.15. The following conditions in a ternary semiring $S$ are equivalent

(i) $S$ is $k$–regular.

(ii) for every bi $k$–ideal $B$ of $S$, $B = BSBSB$.

(iii) for every quasi $k$–ideal $Q$ of $S$, $Q = QSQSQ$.

Proof. (i) $\Rightarrow$ (ii) Let $a \in B$. Since $S$ is $k$–regular, then there exist $x, y \in S$ such that $a + axa = aya$. This implies that $axa + axaxa = aya$. Since $axaxa, aya \in BSBSB$, therefore $axa \in BSBSB$.

Similarly, $aya \in BSBSB$. Therefore, $a \in BSBSB$. Thus, $B \subseteq BSBSB$. Since $B$ is bi $k$–ideal of $S$ therefore $BSBSB \subseteq B$. Hence $B = BSBSB$.

(ii) $\Rightarrow$ (iii) By Result 2.13.

(iii) $\Rightarrow$ (i) Let the condition (iii) holds. Suppose $R$ be right $k$–ideal, $M$ be lateral $k$–ideal and $L$ be left $k$–ideal of $S$, then $Q = R \cap M \cap L$ is a quaisi$k$–ideal of $S$. By hypothesis, we have

\[
R \cap M \cap L = (R \cap M \cap L)S(R \cap M \cap L)S(R \cap M \cap L) \\
\subseteq RSMML \subseteq RML.
\]

But $RML \subseteq R \cap M \cap L$. Therefore, $R \cap M \cap L = RML$. Therefore, by Theorem 2.11, we have $S$ is $k$–regular. □

Theorem 2.16. Let $S$ be a ternary semiring. Then the following are equivalent

(i) $S$ is $k$–regular.

(ii) $B \cap M = BMBMBM$ for every lateral $k$–ideal $M$ and brik–ideal $B$ of $S$.

(iii) $Q \cap M = QMQMQ$ for every lateral $k$–ideal $M$ and quasi$k$–ideal $Q$ of $S$. 
Proof. (i) ⇒ (ii) Let \( a \in B \cap M \). Then \( a \in B \) and \( a \in M \). Since \( S \) is \( k \)--regular then there exist \( x, y \in S \) such that \( a + axa = aya \). This implies that \( axa + axaxa = ayaxa \). Also, \( axaxa + axaxaxa = ayaxaxa \) and \( axaxaxa + axaxaxaxa = ayaxaxaxa \). Since \( axaxaxaxa, ayaxaxaxa, axaxaxaxaxa \in BSMSBSMB \subseteq BMBMB \), therefore by property of \( k \)--closure it is easy to show that \( a \in BMBMB \). Therefore, \( B \cap M \subseteq BMBMB \).

Again, as \( BMBMB \subseteq BSBSB \), therefore \( BMBMB \subseteq BSBSB \subseteq B \) (as \( B \) is bi \( k \)--ideal of \( S \)). Therefore, \( BMBMB \subseteq B \). Also, \( BMBMB \subseteq SM(SSS) \subseteq M, BMBMB \subseteq M = M \) (as \( M \) is lateral \( k \)--ideal of \( S \)). Whence \( BMBMB \subseteq B \cap M \). Therefore, \( B \cap M = \overline{BMBMB} \).

(ii) ⇒ (iii) By Result 2.13, (iii) holds.

(iii) ⇒ (i) Since \( S \) is itself a lateral \( k \)--ideal of \( S \), therefore \( Q = Q \cap S = QSQSQ \). Therefore, by the above theorem the result holds.

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References


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