A NEW CHARACTERIZATION OF $L_2(q)$ WHERE $q = p^n < 125$[1]

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Abstract. It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G. Recently this topic has been finished by V.D. Mazurov, et al. Here the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized $L_2(q)$, where $q = p^n < 125$, by using the order of $L_2(q)$ and the three largest element orders of $L_2(q)$.

Key words: finite group, the largest element order, the second largest element order, the third largest element order, characterization.

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1. Introduction

It is a well-known topic to characterize a finite simple group by using two quantities, the order of G and $\pi_e(G)$ in the past 30 years, where $\pi_e(G)$ denotes the set of orders of elements in G. W.J. Shi characterized some finite simple groups by using $\pi_e(G)$ and $|G|$, for example, see [1]–[6]. Recently, this topic has been finished by V.D. Mazurov, et al. (see [7]). Now, the authors will try to characterize some finite simple groups by using less quantities and have successfully characterized

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simple K_3-groups, sporadic simple groups, $L_3(q)$ and $U_3(q)$, where q are some special powers of primes, by using three numbers: the order of a group, the largest and the second largest element orders in [8]–[10]. In this paper, we characterize $L_2(q)$ for $q = p^n < 125$, via the order of a group and the three largest element orders.

Notations: The groups mentioned are all finite groups, and the number in bracket "( )" behind a group is the order of the group, e.g., $L_2(7)(2^3 \cdot 3 \cdot 7)$ means that $L_2(7)$ is of order $2^3 \cdot 3 \cdot 7$. Let $\pi_e(G)$ denote the set of orders of elements in G, $\pi(G)$, the set of all prime divisors of $|G|$. Let $k_1(G)$ denote the largest element order of G, $k_2(G)$, the second largest element order and $k_3(G)$, the third largest element order. S_p-subgroup is a Sylow p-subgroup of G. We denote by $\Gamma(G)$ the prime graph of G and $t(G)$ is the number of connected components of $\Gamma(G)$. And we also denote the sets of vertex of the connected components of the prime graph by $\{\pi_i, i = 1, \ldots, t(G)\}$. For convenience we call π_i $(1 \leq i \leq t(G))$ the connected components and if the order of G is even, denote the component containing 2 by π_1 (see [11]).

2. Preliminary results

Lemma 1. *Suppose that G has more than one prime graph component. Then one of the following holds:*

(1) *G is a Frobenius group or a 2-Frobenius group;*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.*

Proof. The lemma follows from Theorem A and Lemma 3 in [11].

Remark. A group G will be called a 2-Frobenius group provided G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are *Frobenius* groups with K/H and H as their *Frobenius* kernels respectively.

Lemma 2. *If G is a Frobenius group of even order with K the Frobenius kernel and H the Frobenius complement, then $t(G) = 2$ and $\Gamma(G) = \{\pi(H), \pi(K)\}$* (see [12]).

Lemma 3. *If G is a 2-Frobenius group of even order, then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2$, and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying that $|G/K| \mid |Aut(K/H)|$, $(|G/K|, |K/H|) = 1$, and $|G/K| < |K/H|$. Particularly, G is solvable* (see [12]).

Lemma 4. *Let A be a π'-group of automorphisms of the π-group G, and suppose G or A is solvable. Then for each prime p in π, A leaves invariant some S_p-subgroup of G* (see [13], Theorem 6.22).

Lemma 5. *Let G be $L_2(q)$, where $q = p^n < 125$. Then $|G|$, $k_1(G)$ and $k_2(G)$are as in Table* 1:

Table 1

G	$\|G\|$	$k_1(G)$	$k_2(G)$
$L_2(5) \cong A_5$	$2^2 \cdot 3 \cdot 5$	5	3
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	7	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	9	7
$L_2(9) \cong A_6$	$2^3 \cdot 3^2 \cdot 5$	5	4
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	11	6
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	13	7
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	17	15
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	17	9
$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	19	10
$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	23	12
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	13	12
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	14	13
$L_2(29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	29	15
$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	31	16
$L_2(32)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	33	31
$L_2(37)$	$2^2 \cdot 3^2 \cdot 19 \cdot 37$	37	19
$L_2(41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	41	21
$L_2(43)$	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$	43	22
$L_2(47)$	$2^4 \cdot 3 \cdot 23 \cdot 47$	47	24
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	25	24
$L_2(53)$	$2^2 \cdot 3^3 \cdot 13 \cdot 53$	53	27
$L_2(59)$	$2^2 \cdot 3 \cdot 5 \cdot 29 \cdot 59$	59	30
$L_2(61)$	$2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61$	61	31
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	65	63
$L_2(67)$	$2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 67$	67	34
$L_2(71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	71	36
$L_2(73)$	$2^3 \cdot 3^2 \cdot 37 \cdot 73$	73	37
$L_2(79)$	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 79$	79	40
$L_2(81)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$	41	40
$L_2(83)$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	83	42
$L_2(89)$	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 89$	89	45
$L_2(97)$	$2^5 \cdot 3 \cdot 7^2 \cdot 97$	97	49
$L_2(101)$	$2^2 \cdot 3 \cdot 5^2 \cdot 17 \cdot 101$	101	51
$L_2(103)$	$2^3 \cdot 3 \cdot 13 \cdot 17 \cdot 103$	103	52
$L_2(107)$	$2^2 \cdot 3^2 \cdot 53 \cdot 107$	107	54
$L_2(109)$	$2^2 \cdot 3^3 \cdot 5 \cdot 11 \cdot 109$	109	55
$L_2(113)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$	113	57
$L_2(121)$	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$	61	60

Proof. The lemma follows from [14] and the properties of $L_2(q)$.

Lemma 6. *Let G be a finite group and M be one of the following simple groups: $L_2(5)$, $L_2(9)$, $L_2(11)$, $L_2(13)$, $L_2(16)$, $L_2(17)$, $L_2(19)$, $L_2(23)$, $L_2(25)$, $L_2(27)$, $L_2(29)$, $L_2(32)$, $L_2(37)$, $L_2(41)$, $L_2(43)$, $L_2(47)$, $L_2(53)$, $L_2(59)$, $L_2(61)$, $L_2(67)$, $L_2(71)$, $L_2(73)$, $L_2(79)$, $L_2(81)$, $L_2(83)$, $L_2(89)$, $L_2(97)$, $L_2(101)$, $L_2(103)$, $L_2(107)$, $L_2(109)$, $L_2(113)$, $L_2(121)$.*

Suppose that

(i) $k_1(G) = k_1(M)$;

(ii) $|G| = |M|$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

Proof. We only need to prove the cases $L_2(5)$ $(2^2 \cdot 3 \cdot 5)$, $L_2(53)$ $(2^2 \cdot 3^3 \cdot 13 \cdot 53)$. For the other cases, we can prove them similarly.

1. Assume that $M = L_2(5)$ $(2^2 \cdot 3 \cdot 5)$. In such case, $|G| = 2^2 \cdot 3 \cdot 5$ and $k_1(G) = 5$. Because $k_1(G) = 5$, we get that 5 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Therefore, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

First, we suppose that G is a Frobenius group. Then, by Lemma 2 we get that $t(G) = 2$ and $\Gamma(G) = \{\ \pi(H), \pi(K)\ \}$, where K is the Frobenius kernel and H the Frobenius complement. As $k_1(G) = 5$, K is either a $\{2, 3\}$-Hall subgroup or a Sylow 5-subgroup of G. Since K is nilpotent, let S be a Sylow subgroup of K, one has that $|H| \mid (|S| - 1)$. We can find an suitable Sylow subgroup of K such that $|H| \nmid (|S| - 1)$, and then we get a contradiction. For this reason, K can't be a Sylow 5-subgroup of G. Hence K is a $\{2, 3\}$-Hall subgroup. Consider the Sylow 3-subgroup of K. We can get $5 \mid 2$, a contradiction. Therefore, G is not a Frobenius group.

Second, we suppose that G is a 2-Frobenius group. By Lemma 3, we know that $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 5 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{5\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3\}$ and $|K/H| = 5$. Since $|G/K| \mid |Aut(K/H)| = 4$, we know that $3 \mid |H|$. Consider the action on H by the element of order 5. By Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Since $|L| = 3$, we have $5 \nmid |Aut(L)|$, which means that such action on L is trivial. Therefore, G has an element of order 15, a contradiction. So G is not a 2-Frobenius group.

Remark. This approach can be used to prove that G is not 2-Frobenius group for the most of the other cases. For a few exceptions, we only need to consider $\Omega_1(Z(L))$ of some special Sylow subgroup L to lead to a contradiction. The process can be seen in the case $M = L_2(53)$.

2. Assume that $M = L_2(53)$ $(2^2 \cdot 3^3 \cdot 13 \cdot 53)$. In such case, $|G| = 2^2 \cdot 3^3 \cdot 13 \cdot 53$ and $k_1(G) = 53$. Because $k_1(G) = 53$, we know that 53 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. By Lemma 1, we only need to prove that G is neither a Frobenius group nor a 2-Frobenius group.

Using the similar arguments in case $M = L_2(5)$, we can easily show that G is not a Frobenius group. Now we assert that G is not a 2-Frobenius group. Assume the contrary. Let G be a 2-Frobenius group. By Lemma 3, $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$ and $\pi(G/K) \cup \pi(H) = \pi_1$. Moreover, G/K and K/H are cyclic groups satisfying $|G/K| \mid |Aut(K/H)|$, and $|G/K| < |K/H|$. As 53 is an isolated point of $\Gamma(G)$, $\pi_2(G) = \{53\}$. Therefore, $\pi(G/K) \cup \pi(H) = \{2, 3, 13\}$ and $|K/H| = 53$. Since $|G/K| \mid |Aut(K/H)| = 52$, we know that $3 \mid |H|$. Consider the action on H by the element y of order 53. Again by Lemma 4, there exists a Sylow 3-subgroup L of H fixed by this action. Obviously, $|L| = 3^3$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian 3-group, and $|\Omega_1(Z(L))| \mid 3^3$. Because $\Omega_1(Z(L))$ is characteristic in L, y fixes $\Omega_1(Z(L))$ too. As $53 \nmid |Aut(\Omega_1(Z(L)))|$, the action on $\Omega_1(Z(L))$ by y is trivial, which implies that G has an element of order 159, a contradiction. So G is not a 2-Frobenius group.

3. Main results

Theorem 1. *Let G be a group and M be $L_2(q)$, where $q = p^n \neq 7, 31, 49, 64$, and $q < 125$. Then $G \cong M$ if and only if*

(i) $k_1(G) = k_1(M)$;

(ii) $|G| = |M|$.

Proof. We only need to prove the sufficiency. And the proof will be made through a case by case analysis.

When $q = 5,\ 8,\ 9,\ 17$, the theorem follows from Theorem 1 in [8]. Now, we assume that $q = 11,\ 13,\ 16,\ 19,\ 23,\ 25,\ 27,\ 29,\ 32,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 81,\ 83,\ 89,\ 97,\ 101,\ 103,\ 107,\ 109,\ 113,\ 121$.

Case 1. Since $q = 11,\ 13,\ 16,\ 19,\ 23,\ 25,\ 29,\ 32,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79,\ 81,\ 83,\ 89,\ 97,\ 101,\ 103,\ 107,\ 109,\ 113$ and 121 have similar proofs, we only consider a few of them.

Assume that $q = 11$. In such case, $M = L_2(11)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3 \cdot 5 \cdot 11$ and $k_1(G) = 11$. Therefore, 11 is an isolated point of $\Gamma(G)$ and

$t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $11 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(11)$ $(2^2 \cdot 3 \cdot 5 \cdot 11)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(11)$.

Assume that $q = 13$. In such case, $M = L_2(13)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3 \cdot 7 \cdot 13$ and $k_1(G) = 13$. Therefore, 13 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 7\}$ and $13 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(13)$.

Assume that $q = 16$. In such case, $M = L_2(16)$. By Lemma 5, we know that $|G| = 2^4 \cdot 3 \cdot 5 \cdot 17$ and $k_1(G) = 17$. Therefore, 17 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $17 \in \pi(K/H)$. From [14] we can know that K/H is isomorphic only to $L_2(16)$ $(2^4 \cdot 3 \cdot 5 \cdot 17)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(16)$.

Case 2. $q = 27$. In such case, $M = L_2(27)$. By Lemma 5, we know that $|G| = 2^2 \cdot 3^3 \cdot 7 \cdot 13$ and $k_1(G) = 14$. Therefore, 13 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. So, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 7\}$ and $13 \in \pi(K/H)$. From [14] we can assume that K/H is isomorphic to $L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$ or $L_2(27)$ $(2^2 \cdot 3^3 \cdot 7 \cdot 13)$.

Suppose that $K/H \cong L_2(13)$ $(2^2 \cdot 3 \cdot 7 \cdot 13)$. From [14] we know that $3 \nmid |Out(K/H))| = 2$, so $3 \mid |H|$ and $|H| = 3^2$. Consider the action on H by the element of order 13. Clearly, this action is trivial, which implies that G has an element of order 39, a contradiction.

Therefore, we have $K/H \cong L_2(27)$ $(2^2 \cdot 3^3 \cdot 7 \cdot 13)$. So $H = 1$, $K = G$, and therefore $G \cong L_2(27)$. This completes the proof.

Theorem 2. *Let G be a group and M be $L_2(q)$ for $q = 7, 31$. Then $G \cong M$ if and only if*

(i) $k_i(G) = k_i(M)$, *where* $i = 1, 2$

(ii) $|G| = |M|$.

Proof. It is enough to prove the sufficiency. Firstly, we suppose that $q = 7$. In such case, the theorem follows from the theorem 2 in [7]. Now, we assume that $q = 31$. In this case, $M = L_2(31)$ $(2^5 \cdot 3 \cdot 5 \cdot 31)$, and therefore $|G| = 2^5 \cdot 3 \cdot 5 \cdot 31$, $k_1(G) = 31$ and $k_2(G) = 16$. At first, we show that G is a non-solvable group. Assume the contrary. Let G be a solvable group. Then the minimal normal subgroup N of G is an elementary abelian p-group. If N is a 3-group, then $|N| = 3$ and $|Aut(N)| = 2$. Consider the action on N by an element of order 31. Obviously, this action is trivial. Therefore, G has an element of order 93, a contradiction. If N is a 5-group, then $|N| = 5$ and $|Aut(N)| = 4$. Consider the action on N by an element of order 31. Clearly, this action is also trivial. Therefore, G has an element of order 155, also a contradiction. If N is a 2-group, then $|N \mid |2^5$. Suppose that $|N| = 2^5$. Then N has an element with order 16 for $k_2(G) = 16$, which contradicts that N is an elementary abelian 2-group. So, $|N| \mid 2^4$, and hence $31 \nmid |Aut(N)|$. Consider the action on N by the element of order 31. We can see that this action is trivial. Therefore, G has an element of order 62, still a contradiction. Now assume that N is a 31-group. Then $|N| = 31$. Let $N =< a >$. Since $k_2(G) = 16$, G must have an element of order 4. Consider the action on N by an element x of order 4. There must exist a positive integer t such that $a^{x^t} = a$. Because $|Aut(N)| = 30$, we can conclude that $t = 1$, 2 or 3. If $t = 1$ or 3, then $|x^t| = 4$. In this case, G has an element of order 124, a contradiction. If $t = 2$, then $|x^t| = 2$. In such case, G has an element of order 62, also a contradiction. Therefore, G is a non-solvable group.

Because $k_1(G) = 31$, 31 is an isolated point of $\Gamma(G)$ and $t(G) \geq 2$. By Lemma 1, we know that G is either a Frobenius group or a 2-Frobenius group, or has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$.

First, we assume that G is a Frobenius group. By Lemma 2, we get that $t(G) = 2$ and $\Gamma(G) = \{\ \pi(H),\ \pi(K)\ \}$, where K is the Frobenius kernel and H the Frobenius complement. Since $k_1(G) = 31$, K is either a $\{2,\ 3,\ 5\}$-Hall subgroup or a Sylow 31-subgroup of G. If K is a Sylow 31-subgroup of G, then $|K| = 31$. Consider the action on K by an element x of order 4. Using the similar discussion in preceding paragraph, we can get that G has an element of order larger than 31, which is a contradiction. So we suppose that K is a $\{2,\ 3,\ 5\}$-Hall subgroup. Consider the action on K by the element of order 31. By Lemma 4 we can draw a conclusion that there exists a Sylow 5-subgroup L of K fixed by this action. Since $G = 2^5 \cdot 3 \cdot 5 \cdot 31$, we have $|L| = 5$ and thus $31 \nmid Aut(L)$, which implies that G has an element of order 155, a contradiction too. Therefore, G is not a Frobenius group.

Now we assume that G is a 2-Frobenius group. By Lemma 3, we know that G is solvable, also a contradiction. So G is not a 2-Frobenius group.

Therefore, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^5 \cdot 3 \cdot 5 \cdot 31$, and 31 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq$

$\{2, 3, 5\}$ and $31 \in \pi(K/H)$. From [14] we know that K/H is isomorphic only to $L_2(31)(2^5 \cdot 3 \cdot 5 \cdot 31)$. Therefore, $H = 1$ and $K = G$, and thus $G \cong L_2(31)$. The proof is completed. ∎

Let $k_3(G)$ denote the third largest element order of G. We can easily know that $k_3(L_2(49)) = 12$ and $k_3(L_2(64)) = 21$ from the properties of $L_2(q)$.

For $q = 49$ and 64, we have the following result.

Theorem 3. *Let G be a group and M be $L_2(q)$ for $q = 49, 64$. Then $G \cong M$ if and only if*

(i) $k_i(G) = k_i(M)$, *where* $i = 1, 2, 3$

(ii) $|G| = |M|$.

Proof. We only need to prove the sufficiency.

Case 1. Assume that $M = L_2(49)$ $(2^4{\cdot}3{\cdot}5^2{\cdot}7^2)$. In such case, $|G| = 2^4{\cdot}3{\cdot}5^2{\cdot}7^2$ and $k_1(G) = 25$, $k_2(G) = 24$, $k_3(G)=12$, from which we know that 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar discussion in Lemma 6, we can prove that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and 7 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $7 \in \pi(K/H)$. From [14] we can suppose that $K/H \cong L_2(7)$ $(2^3 \cdot 3 \cdot 7)$ or $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$.

First, we assume that $K/H \cong L_2(7)$. From [14] we know that $|Out(L_2(7))|=2$ and thus $|G/K| | 2$. Therefore, $7 \mid |H|$. Let L be a Sylow 7-subgroup of H. Then $|L| = 7$. As H is a nilpotent group, we have L is characteristic in H, and thus $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 35, a contradiction.

Therefore, we can get that $K/H \cong L_2(49)$. Because $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, we can conclude that $H = 1$ and $K = G$, and thus $G \cong L_2(49)$.

Case 2. Assume that $M = L_2(64)$ $(2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13)$. In such case, $|G| = 2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$ and $k_1(G) = 65$, $k_2(G) = 63$, $k_3(G)=21$, from which we know that 7 is an isolated point of $\Gamma(G)$, and therefore $t(G) \geq 2$. Using the similar discussion in Lemma 6, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/K| \mid |Out(K/H)|$. Because $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and 7 is an isolated point of $\Gamma(G)$, we have $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 13\}$ and $7 \in \pi(K/H)$. From [14] we can suppose that K/H is isomorphic to one of the following simple groups: $L_2(7)$ $(2^3 \cdot 3 \cdot 7)$, $L_2(8)$ $(2^3{\cdot}3^2{\cdot}7)$, A_7 $(2^3{\cdot}3^2{\cdot}5{\cdot}7)$, A_8 $(2^6{\cdot}3^2{\cdot}5{\cdot}7)$, $L_3(4)(2^6{\cdot}3^2{\cdot}5{\cdot}7)$, $Sz(8)$ $(2^6{\cdot}5{\cdot}7{\cdot}13)$ and $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$.

Suppose that K/H is isomorphic to one of the simple groups mentioned above, except $Sz(8)$ $(2^6 \cdot 5 \cdot 7 \cdot 13)$, $L_2(64)$ $(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$. From [14] we know that $13 \nmid |Out(K/H)|$ and thus $13 \mid |H|$. Let L be a Sylow 13-subgroup of H. Then

$|L| = 13$ and $L \trianglelefteq G$. Consider the action on L by the element of order 5. Clearly, this action is trivial. It implies that G has an element of order 65, a contradiction.

Suppose that $K/H \cong Sz(8)$. From [14] we know that $|Out(Sz(8))| = 3$ and hence $|G/K||3$. Therefore, we can get that $3 \mid |H|$ by comparing the order of G. Let L be a Sylow 3-subgroup of H. Then $|L| \mid 3^2$ and $L \trianglelefteq G$. Consider the action on L by the element of order 13. Clearly, this action is trivial. It implies that G has an element of order 39, a contradiction.

Therefore, we have $K/H \cong L_2(64)$. Because $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, we can get that $H = 1$ and $K = G$, and thus $G \cong L_2(64)$. This completes the proof. ∎

As a corollary of the proceeding theorems, we have

Theorem 4. *Let G be $L_2(q)$, where $q = p^n < 125$. Then G can be uniquely determined by the order of G and $k_i(G)$, where $i \leq 3$.*

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