AN APPROACH TO THE APPROXIMATION OF THE INVERSE OF A SQUARE MATRIX BY HE’S HOMOTOPY PERTURBATION METHOD

B. Keramati

Department of Mathematics
Faculty of Science
Semnan University
Sadi square, Semnan
P.O.Box 35195-363
Iran
e-mail: br_keramati@yahoo.com

Abstract. In this paper, we present an efficient numerical algorithm for approximating the inverse of a square matrix based on homotopy perturbation method. Some numerical illustrations are given to show the efficiency of the algorithm.

1. Introduction


This method has been successfully applied to solve many types of nonlinear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called “homotopy perturbation method” (HPM), is provided by He [7]–[10], as well as the recent developments[4]–[6].

In most cases, using HPM, gives a very rapid convergence of the solution series, and usually only a few iterations leading to very accurate solutions, specially when the modified one is applied [1], [17].

In this article, HPM is applied to the equation $AX - I = 0$ and the convergence of the method is considered under certain conditions.

2. Analysis of the method

Consider the equation

$$AX - I = 0,$$ (1)
where 
\[ A = [a_{ij}], \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n. \]

Let 
\[ L(U) = AU - I, \]
\[ F(U) = U - W_0, \]

we define homotopy \( H(U, p) \) by
\[ H(U, 0) = F(U), \]
\[ H(U, 1) = L(U), \]

and typically, a convex homotopy as follows
\[ H(U, p) = (1 - p)F(U) + pL(U) = 0, \quad (2) \]

where \( F \) is an operator with known solution \( W_0 \). HPM uses the homotopy parameter \( p \) as an expanding parameter\[4,6\] to obtain
\[ U = U_0 + pU_1 + p^2U_2 + \cdots, \quad (3) \]

and for \( p \to 1 \), it gives an approximation to the solution of (1) as follows
\[ V = \lim_{p \to 1} (U_0 + pU_1 + p^2U_2 + \cdots). \]

By substituting (3) in (2) and equating the terms with identical powers of \( p \), we obtain
\[ p^0: \quad U_0 - W_0 = 0, \quad U_0 = W_0, \]
\[ p^1: \quad (A - I)U_0 + U_1 - W_0 - I = 0, \quad U_1 = I - (A - I)U_0 + W_0, \]
\[ p^2: \quad (A - I)U_1 + U_2 = 0, \quad U_2 = -(A - I)U_1 \]

and in general
\[ U_{n+1} = -(A - I)U_n, \quad n = 1, 2, \cdots, \]

if we take \( U_0 = W_0 = 0 \), then we have
\[ U_1 = I, \]
\[ U_2 = -(A - I)U_1, \]
\[ = (A - I), \]
\[ U_3 = (A - I)^2, \]
\[ \vdots \]
\[ U_{n+1} = (-1)^n(A - I)^n, \]
hence, the solution can be of the form
\[ U = U_0 + U_1 + U_2 + \cdots, \]
or
\[ U = [I - (A - I) + (A - I)^2 - \cdots]. \tag{4} \]

**Theorem 1.** The sequence
\[ U^{[m]} = \left[ \sum_{k=0}^{m} (A - I)^k \right], \]
is a Cauchy sequence if
\[ \| A - I \| < 1. \]

**Proof.** We must show that
\[ \lim_{m \to \infty} \| U^{[m+p]} - U^{[m]} \| = 0, \]
so, for showing this we can write
\[ U^{[m+p]} - U^{[m]} = \sum_{k=1}^{p} (-1)^{m+k} (A - I)^{m+k}, \]
or
\[ \| U^{[m+p]} - U^{[m]} \| \leq \sum_{k=1}^{p} \| A - I \|^{m+k}, \]
let \( \gamma = \| A - I \|, \) then
\[ \| U^{[m+p]} - U^{[m]} \| \leq \gamma^m \sum_{k=1}^{p} \gamma^k, \]
\[ \leq \left( \frac{\gamma^p - 1}{\gamma - 1} \right) \gamma^m, \]
now if \( \gamma < 1, \) then we have
\[ \lim_{m \to \infty} \| U^{[m+p]} - U^{[m]} \| \leq \left( \frac{\gamma^p - 1}{\gamma - 1} \right) \left( \lim_{m \to \infty} \gamma^m \right), \]
therefore, we obtain
\[ \lim_{m \to \infty} \| U^{[m+p]} - U^{[m]} \| = 0, \]
which completes the proof. \( \blacksquare \)
Lemma 1. If $A$ is diagonally dominated and

$$D = \text{diag} \left[ \frac{1}{a_{ii}} \right],$$

then

$$\|DA - I\|_\infty < 1.$$  

Proof. Let $C = DA - I$, then it can be easily shown that

$$c_{ij} = \begin{cases} 0 & i = j \\ \frac{a_{ij}}{a_{ii}} & i \neq j, \end{cases}$$

since $A$ is diagonally dominated, then

$$|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|,$$

or

$$\sum_{j=1,j\neq i}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| < 1, \quad i = 1, 2, \cdots n,$$

hence

$$\sum_{j=1}^{n} |c_{ij}| < 1, \quad i = 1, 2 \cdots n,$$

which implies

$$\| C \|_\infty = \| DA - I \|_\infty < 1.$$

If $\| A - I \|_\infty > 1$, by pre-multiplying both sides of equation (1) by matrix $D$, we rewrite the equation as follows

$$DAX - D = 0$$

and it can be easily shown that

$$U_{n+1} = (-1)^n(DA - I)^nD$$  \hspace{1cm} (5)

3. Numerical results

Example 1. Approximate the inverse of the matrix
an approach to the approximation of the inverse of a square matrix...

\[ A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 1 \\ 0 & 1 & -3 \end{bmatrix}. \]

Since \( \| A - I \|_\infty = 8 \), and \( A \) is diagonally dominated, we can write

\[ D = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}, \]

hence

\[ DA = \begin{bmatrix} 1 & 1/4 & -1/4 \\ -1/6 & 1 & 1/6 \\ 0 & -1/3 & 1 \end{bmatrix}, \]

\[ DA - I = \begin{bmatrix} 0 & 1/4 & -1/4 \\ -1/6 & 0 & 1/6 \\ 0 & -1/3 & 0 \end{bmatrix}, \]

from (4) and (5) we have

\[ U = [I - (DA - I) + (DA - I)^2 - \cdots]D, \]

and using six terms, we approximate \( A^{-1} = U \) as follows

\[ U \approx U_0 + U_1 + \cdots + U_5, \]

or

\[ U \approx \begin{bmatrix} 0.2435 & -0.025 & -0.089 \\ 0.0385 & 0.1539 & 0.0387 \\ 0.0127 & 0.0514 & -0.32 \end{bmatrix}, \]

where the exact solution is
Example 2. Give an approximation to the inverse of the following matrix

\[
A = \begin{bmatrix}
0.5 & 0.5 & 0.2 \\
0.1 & 0.3 & 0.1 \\
0.1 & 0.1 & 0.3
\end{bmatrix}
\]

Matrix \( A \) is nearly diagonally dominated, so by similar operations we obtain

\[
DA - I = \begin{bmatrix}
0 & 1 & 0.4 \\
1 & 0 & 1 \\
\frac{1}{3} & 1 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0
\end{bmatrix},
\]

and using seven iterations, an approximation to \( A^{-1} \) will be as follows

\[
\begin{bmatrix}
3.1956 & -4.4739 & -1.5056 \\
-0.66063 & 5.19744 & -0.97097 \\
-0.66063 & 0.19973 & 4.02674
\end{bmatrix},
\]

where the exact solution is

\[
A^{-1} = \begin{bmatrix}
3.07692 & -5 & -0.38462 \\
-0.76923 & 5 & -1.15385 \\
-0.76923 & 0 & 3.84615
\end{bmatrix}.
\]

4. Conclusion

In this paper, we used homotopy perturbation method to approximate the inverse of a diagonally dominated matrix in terms of the subtraction of the given matrix (or its modification) and unit matrix. Solved problems show the convergence of the method increases as the the matrix becomes more strictly diagonally dominated.
References


Accepted: 16.04.2009