ON SOME CLASSES OF SUBMANIFOLDS SATISFYING CHEN'S EQUALITY IN AN EUCLIDEAN SPACE

Cihan Özgür

Department of Mathematics
Balıkesir University
10145, Çağış, Balıkesir
Turkey
e-mail: cozgur@balikesir.edu.tr

Uday Chand De

Department of Pure Mathematics
University of Calcutta
35, B.C. Road, Kolkata-700019
India
e-mail: uc_de@yahoo.com

Abstract. We study submanifolds satisfying Chen's equality in an Euclidean space. Firstly, we consider projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also study submanifolds satisfying the condition $P \cdot P = 0$.

Keywords: Chen invariant, Chen's inequality, projectively semi-symmetric manifold, totally geodesic submanifold, minimal submanifold.
2000 Mathematics Subject Classification: 53C40, 53C25, 53C42.

1. Introduction

One of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. In [1] and [4], B.Y. Chen established inequalities in this respect, called Chen inequalities. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature and the Ricci curvature; and the well known modern curvature invariant namely Chen invariant [2]. In 1993, Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications. This inequality is now well known as Chen's inequality; and in the equality case it is known as Chen's equality.

In [6], Dillen, Petrovic and Verstraelen studied Einstein, conformally flat and semisymmetric submanifolds satisfying Chen's equality in Euclidean spaces. In [8], the first author and M.M. Tripathi studied the same problems for a submanifold of a real space form.

In this paper, we study submanifolds satisfying Chen's equality and the conditions $R \cdot P = 0$ and $P \cdot P = 0$ in an Euclidean space.

The paper is organized as follows. In Section 2, we give some known results about Riemannian submanifolds and Chen's inequality which will be used in the next sections. In Section 3, we study projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also study submanifolds satisfying the condition $P \cdot P = 0$.

2. Chen's inequality

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Euclidean space $\mathbb{E}^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$R(X,Y,Z,W) = \langle \sigma(X,W), \sigma(Y,Z) \rangle - \langle \sigma(X,Z), \sigma(Y,W) \rangle \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where R is the curvature tensors of M.

The mean curvature vector H is given by $H = \frac{1}{n}\,\mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\mathbb{E}^{m+n}$ if $\sigma = 0$, and *minimal* if $H = 0$ [3].

Let $\{e_1, ..., e_n\}$ be an orthonormal tangent frame field on M. For the plane section $e_i \wedge e_j$ of the tangent bundle TM spanned by the vectors e_i and e_j $(i \neq j)$ the scalar curvature of M is defined by $\kappa = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$ where K denotes the sectional curvature of M. Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{K(\pi) \mid \pi \text{ is a plane in } T_x M^n\}.$$

Note that since the set of planes at a certain point is compact, this infimum is actually a minimum.

Lemma 2.1. [1] *Let M, $n \geq 2$, be any submanifold of $\mathbb{E}^{n+m}$. Then*

$$\inf K \geq \frac{1}{2}\left\{\kappa - \frac{n^2(n-2)}{n-1}\,|H|^2\right\}. \tag{2.2}$$

Equality holds in (2.2) *at a point x if and only if with respect to suitable local orthonormal frames $e_1, ..., e_n \in T_xM^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$, $t = 1, ..., p$ are given by*

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{bmatrix},$$

$$A_t = \begin{bmatrix} c_t & d_t & 0 & \cdots & 0 \\ d_t & -c_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad (t > 1), \tag{2.3}$$

where $\mu = a + b$ for any such frame, $\inf K(x)$ *is attained by the plane $e_1 \wedge e_2$.*

The inequality (2.2) is well known as Chen's inequality. In case of equality, it is known as Chen's equality. For dimension $n = 2$, the Chen's equality is always true.

Let M be an n-dimensional ($n \geq 3$) submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. Then, from Lemma 2.1 we immediately have the following

$$K_{12} = ab - \sum_{r=1}^{m}(c_r^2 + d_r^2), \tag{2.4}$$

$$K_{1j} = a\mu, \tag{2.5}$$

$$K_{2j} = b\mu, \tag{2.6}$$

$$K_{ij} = \mu^2, \tag{2.7}$$

$$S(e_1, e_1) = K_{12} + (n-2)a\mu, \tag{2.8}$$

$$S(e_2, e_2) = K_{12} + (n-2)b\mu, \tag{2.9}$$

$$S(e_i, e_i) = (n-2)\mu^2, \tag{2.10}$$

where $i, j > 2$. Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different [6].

Projective curvature tensor of submanifolds satisfying Chen's equality

In this section, we consider projectively semi-symmetric submanifolds satisfying Chen's equality in an Euclidean space. We also consider submanifolds satisfying the condition $P \cdot P = 0$.

The *projective curvature tensor* P *of an* n*-dimensional Riemannian manifold* (M, g) is defined by [9]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{3.1}$$

It is well-known that if the condition $R \cdot P = 0$ holds on M, then M is said to be *projectively semi-symmetric*.

So from (2.4)-(2.10) we have the following corollary:

Corollary 3.1. *Let* M *be an* n*-dimensional* $(n \geq 3)$ *submanifold in an Euclidean space satisfying Chen's equality, then*

$$P_{122} = \frac{n-2}{n-1}\left(K_{12} - b\mu)\right) e_1, \tag{3.2}$$

$$P_{133} = \mu\left(a - \frac{n-2}{n-1}\mu\right) e_1, \tag{3.3}$$

$$P_{131} = \frac{1}{n-1}\left(K_{12} - a\mu\right) e_{3,} \tag{3.4}$$

$$P_{233} = \mu\left(b - \frac{n-2}{n-1}\mu\right) e_2, \tag{3.5}$$

$$P_{211} = \frac{n-2}{n-1}\left(K_{12} - a\mu\right) e_2 \tag{3.6}$$

$$P_{232} = \frac{1}{n-1}\left(K_{12} - b\mu\right) e_{3,} \tag{3.7}$$

and

$$P_{ijk} = 0 \text{ if } i, j, k \text{ are mutually different.} \tag{3.8}$$

Theorem 3.2. *Let* M *be an* n*-dimensional* $(n \geq 3)$ *submanifold of an Euclidean space* $\mathbb{E}^{n+m}$ *satisfying Chen's equality. If* M *is projectively semi-symmetric then*

(i) M *is totally geodesic, or*

(ii) M *is minimal, or*

(iii) *M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, or*

(iv) *$infK = 0$, or*

(v) *$a = b$, in this case if $n = 3$ then M is totally geodesic, if $n = 4$ then M is a pseudosymmetric hypersurface of $\mathbb{E}^5$ which has a shape operator of the form*

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}, \tag{3.9}$$

or

(vi) *M is a submanifold in some totally geodesic subspace $\mathbb{E}^{n+m-1}$ which has shape operators of the form (2.3).*

Proof. Assume that the condition $R \cdot P = 0$ holds on M. Then, we can write

$$\begin{aligned} (R(e_1,e_3)\cdot P)(e_2,e_3,e_1) &= R(e_1,e_3)P(e_2,e_3)e_1 \\ &-P(R(e_1,e_3)e_2,e_3)e_1 - P(e_2,R(e_1,e_3)e_3)e_1 \\ &-P(e_2,e_3)R(e_1,e_3)e_1 = 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} (R(e_2,e_3)\cdot P)(e_1,e_3,e_2) &= R(e_2,e_3)P(e_1,e_3)e_2 \\ &-P(R(e_2,e_3)e_1,e_3)e_2 - P(e_1,R(e_2,e_3)e_3)e_2 \\ &-P(e_1,e_3)R(e_2,e_3)e_2 = 0. \end{aligned} \tag{3.11}$$

Then, using (2.4)-(2.7) and (3.2)-(3.8), we get

$$a\mu\left[b\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2)\right] = 0 \tag{3.12}$$

and

$$b\mu\left[a\mu - (n-2)ab + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2)\right] = 0. \tag{3.13}$$

Case I. If M is totally geodesic, the condition $R \cdot P = 0$ holds trivially.

Case II. If $\mu = 0$ then M is minimal.

Case III. If $\mu \neq 0$ and $a = 0$ then $\mu = b$. Hence, from (3.13), we get $(n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0$. This gives us $c_r = d_r = 0$. So, by [5], M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$.

Case IV. If $\mu \neq 0$ and $b = 0$, then we obtain again the same result in Case III.

Case V. $a, b, \mu \neq 0$, then from (3.12) and (3.13) we obtain $a = b$, or $\mu = 0$, or $K_{12} = 0$. If $\mu = 0$, then M is minimal. If $K_{12} = 0$ then $\inf K = 0$. Assume that $a = b$. Then, from (3.13) we have

$$(4-n)a^2 + (n-2)\sum_{r=1}^{m}(c_r^2 + d_r^2) = 0.$$

In this case, if $n = 3$, then $c_r = d_r = 0$. Hence, M is totally geodesic. If $n = 4$, then $c_r = d_r = 0$, so by Theorem 2.12 of [7], M is a pseudosymmetric hypersurface in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$, which has a shape operator of the form (3.9).

Case VI. If $a = b = 0$, then M is a submanifold in some totally geodesic subspace $\mathbb{E}^{n+m-1}$, which has shape operators of the form (2.3).

This completes the proof of the theorem. ∎

Theorem 3.3. *Let M be an n-dimensional ($n \geq 3$) submanifold of an Euclidean space $\mathbb{E}^{n+m}$ satisfying Chen's equality. If the condition $P \cdot P = 0$ holds on M, then*

(i) *M is minimal, or*

(ii) *M is totally geodesic, or*

(iii) *M is 3-dimensional, or*

(iv) *$a = b$.*

Proof. Since the condition $P \cdot P = 0$ holds on M we have

$$\begin{aligned}(P(e_1,e_3)\cdot P)(e_2,e_3,e_1) &= P(e_1,e_3)P(e_2,e_3)e_1 \\ &\quad - P(P(e_1,e_3)e_2,e_3)e_1 - P(e_2,P(e_1,e_3)e_3)e_1 \\ &\quad - P(e_2,e_3)P(e_1,e_3)e_1 = 0\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}(P(e_2,e_3)\cdot P)(e_1,e_3,e_2) &= P(e_2,e_3)P(e_1,e_3)e_2 \\ &\quad - P(P(e_2,e_3)e_1,e_3)e_2 - P(e_1,P(e_2,e_3)e_3)e_2 \\ &\quad - P(e_1,e_3)P(e_2,e_3)e_2 = 0.\end{aligned} \tag{3.15}$$

So, in view of (3.2)-(3.8), we obtain

$$\mu\left[K_{12} - a\mu\right]\left[\left(a - \frac{n-2}{n-1}\mu\right)(n-2) + \left(b - \frac{n-2}{n-1}\mu\right)\right] = 0 \tag{3.16}$$

and

$$\mu\,[K_{12} - b\mu]\left[\left(b - \frac{n-2}{n-1}\mu\right)(n-2) + \left(a - \frac{n-2}{n-1}\mu\right)\right] = 0 \tag{3.17}$$

Case I. If $\mu = 0$, then M is minimal.

Case II. If $K_{12} - a\mu = 0$, $K_{12} - b\mu = 0$, then we obtain $a = b$. Since $K_{12} - a\mu = 0$, from (2.4) we get $a^2 + \sum_{r=1}^{m}(c_r^2 + d_r^2) = 0$, which gives us $a = c_r = d_r = 0$. Hence, M is totally geodesic.

Case III. If

$$\left(a - \frac{n-2}{n-1}\mu\right)(n-2) + \left(b - \frac{n-2}{n-1}\mu\right) = 0$$

$$\left(b - \frac{n-2}{n-1}\mu\right)(n-2) + \left(a - \frac{n-2}{n-1}\mu\right) = 0,$$

and $\mu \neq 0$ and $K_{12} - a\mu \neq 0$, then either $n = 3$ or $a = b$.

Case IV. If

$$K_{12} - a\mu = 0 \text{ and } \left(b - \tfrac{n-2}{n-1}\mu\right)(n-2) + \left(a - \tfrac{n-2}{n-1}\mu\right) = 0,$$

then, from (2.4), we get $a = c_r = d_r = 0$, which gives us $a = c_r = d_r = 0$. Hence, M is totally geodesic.

Case V. If

$$K_{12} - b\mu = 0 \text{ and } \left(a - \tfrac{n-2}{n-1}\mu\right)(n-2) + \left(b - \tfrac{n-2}{n-1}\mu\right) = 0,$$

then, from (2.4), we get $a = b = c_r = d_r = 0$, which gives us M is totally geodesic.

This proves the theorem. ∎

References

[1] Chen, B.-Y., *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel), 60 (6) (1993), 568–578.

[2] Chen, B.-Y., *A Riemannian invariant for submanifolds in space forms and its applications*, in Geometry and Topology of Submanifolds VI, World Scientific, Singapore (1994), 58–81.

[3] Chen, B.-Y., *Riemannian submanifolds*, in Handbook of Differential Geometry, vol. I, F. Dillen and L. Verstraelen (eds.), North Holland, Amsterdam, 2000, 187–418.

[4] Chen, B.-Y., *δ-invariants, Inequalities of Submanifolds and Their Applications*, in Topics in Differential Geometry, A. Mihai, I. Mihai, R. Miron (eds.), Editura Academiei Romane, Bucuresti, 2008, 29–156.

[5] Deprez, J., *Semi-parallel immersions*, Geometry and Topology of Submanifolds (Marseille, 1987), 73–88, World Sci. Publ., Teaneck, NJ, 1989.

[6] Dillen, F., Petrovic, M. and Verstraelen, L., *Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality*, Israel J. Math., 100 (1997), 163-169.

[7] Özgür, C. and Arslan, K., *On some class of hypersurfaces in $\mathbb{E}^{n+1}$ satisfying Chen's equality*, Turkish J. Math., 26 (2002), 283-293.

[8] Özgür, C. and Tripathi, M.M., *On submanifolds satisfying Chen's equality in a real space form*, Arabian Journal for Science and Engineering, 33 (2A) (2008), 321-330.

[9] Yano, K. and Kon, M., *Structures on manifolds*, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.

Accepted: 15.04.2009