DEGENERACY OF SOME CLUSTER SETS

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Abstract. Using θ-closure [39], δ-closure [39] of a subset and β-open [1] sets, we initiate two new kinds of cluster sets for functions and multifunctions. An explicit expression of each kind of cluster sets are given in terms of filters and grills [38] and also several of their properties are investigated. In the process, the degeneracy of such cluster sets are used as tools to obtain new characterizations of various separation axioms. As application, such investigations ultimately provide new techniques for studying the covering property β-closedness [7].subjclass[2000]Primary 54D20, 54D30, 54C60, 54C99.

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1. Introduction

Cluster sets for functions and multifunctions in general topology have been investigated recently by good many researchers although such notion was developed long ago and studied extensively within the framework of real and analytic function theory (see [42]). In this regard, the classical book of Collingwood and Lohwater [12] and the research papers of Weston [40] and Hunter [20] are worth to be mentioned where a comprehensive collections of works are found. Subsequently, Hamlett [18], [19], Joseph [22] and many others (see [26], [27]) have extended this notion for investigation of various characterizations of different covering properties like compactness, Lindeloffness, H-closedness [39], minimal Hausdorffness, P-closedness [3], S-closedness [37].

Since the introduction of β-open sets [1] (= semipreopen [4]), many topological concepts are studied in terms of β-open sets. Only very few we mention which are found in the papers [1, 2, 4, 5, 6, 7, 8, 11, 16, 17, 21, 30, 31, 32, 33]. In this paper, using β-open sets and θ-closure [39] (resp. δ-closure [39]) due to Veličko [39], we introduce two new kinds of cluster sets called β-θ-cluster (resp. β-δ-cluster) sets which provide new techniques for the study of the covering property β-closedness [7]. (A space X is called β-closed if for every covering of X by β-open sets has finite subfamily whose β-closures cover X.) In the course of study, the degeneracy of such cluster sets are used as tools to obtain new characterizations of various separation axioms like Hausdorffness, weakly Hausdorffness, Urysohnness and other relevant properties.

Throughout the paper, X and Y denote topological spaces without any separation axioms, and $\psi : X \rightarrow Y$ denotes a single valued function from X into Y. By a multifunction $F : X \rightarrow Y$, we mean a function mapping points of X into the nonempty subsets of Y. For a subset S of X, clS and $intS$ represent the closure of S and interior of S in X respectively. We recall the following well known definitions: A subset S of a space (X, τ) or X is said to be α-open [28] (resp. semi-open [24], pre-open [25], β-open [1] or semi-preopen [4]) if $S \subset intclintS$ (resp. $S \subset clintS$, $S \subset intclS$ or $S \subset clintclS$). We denote the classes of all α-open (resp. semi-open, pre-open, β-open or semi-preopen) sets in a space X by τ_α (resp. $SO(X), PO(X), \beta O(X) = SPO(X)$). It is well known that $\tau \subset \tau_\alpha = PO(X) \cap SO(X) \subset PO(X) \cup SO(X) \subset \beta O(X)$. The family of all open (resp. β-open) sets containing a subset S is denoted by $\tau(S)$ (resp. $\beta O(X, S)$). If in particular $S = \{x\}$ then they are respectively denoted by $\tau(x)$ and $\beta O(X, x)$. The complement of a β-open set is called a β-closed set. Pre-closed and semi-closed sets are defined similarly. The β-closure=sp-closure [4] (resp. pre-closure, semi-closure) of S denoted by βclS (resp. $pclS$, $sclS$) is the intersection of all β-closed (resp. pre-closed and semi-closed) subsets of X containing S. A space X is quasi-H-closed [10] (resp. S-closed [37], P-closed [3]) if every open (resp. semi-open, pre-open) cover of X has finite subfamily whose closures (resp. closures, pre-closures) cover X. For a subset S of X, the θ-closure [39] (resp. δ-closure [39]) of S, denoted by θ-clS (resp. δ-clS) is the set $\{x \in X : clU \cap S \neq \emptyset$ for each $U \in \tau(x)\}$ (resp. $\{x \in X : intclU \cap S \neq \emptyset)$ for each $U \in \tau(x)\}$). S is called

θ-closed (resp. δ-closed) if $S = \theta\text{-}clS$ (resp. $S = \delta\text{-}clS$). A space X is called weakly Hausdorff [36], if each point x of X is the intersection of all regular closed sets containing x.

A filter base $\mathcal{F}$ on a space X is said to be β-θ-adhere [7] at x if $F \cap \beta clU \neq \emptyset$, for each $F \in \mathcal{F}$ and $U \in \beta O(X, x)$. Thron [38] has defined a non-empty family $\mathcal{G}$ of non-empty subsets of X to be a grill if (i) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$ and (ii) $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$. A grill $\mathcal{G}$ on X is said to β-θ-converges to a point x of (X, τ), if for each $U \in \beta O(X, x)$ there is a $G \in \mathcal{G}$ with $G \subseteq \beta clU$. A subset A of a space X is said to be NC-set [35] if for every cover of A by means of regular open or δ-open sets of X has a finite subcover.

2. Prerequisites

The following definitions and results will be frequently used in the subsequent sections.

Definition 2.1 A subset S of a space (X, τ) is said to be β-regular (= semipre-regular [30]) if it is both β-open as well as β-closed.

The family of all β-regular sets of a space X and that containing a point x of X are respectively denoted by $\beta R(X)$ and $\beta R(X, x)$.

Lemma 2.2 [30] *For a subset S of a space X, $S \in \beta O(X)$ if and only if $\beta clS \in \beta R(X)$.*

Definition 2.3 A point $x \in X$ is said to be in the β-θ-closure (= sp-θ-closure [30]) of S, denoted by $\beta\text{-}\theta\text{-}cl(S)$, if $S \cap \beta clV \neq \emptyset$ for every $V \in \beta O(X, x)$. If $\beta\text{-}\theta\text{-}clS = S$, then S is said to be β-θ-closed (=sp-θ-closed [30]). The complement of a β-θ-closed set is said to be β-θ-open (=sp-θ-open [30]).

Lemma 2.4 [30] *For a subset A of a space X, $\beta\text{-}\theta\text{-}cl(A) = \cap\{R : A \subset R$ and $R \in \beta R(X)\}$.*

Lemma 2.5 [30] *Let A and B be any subsets of a space X. Then the following properties hold:*

(i) *$x \in \beta\text{-}\theta\text{-}cl(A)$ if and only if $A \cap V \neq \emptyset$ for each $V \in \beta R(X, x)$,*

(ii) *if $A \subset B$ then $\beta\text{-}\theta\text{-}clA \subset \beta\text{-}\theta\text{-}clB$,*

(iii) *$\beta\text{-}\theta\text{-}cl(\beta\text{-}\theta\text{-}clA) = \beta\text{-}\theta\text{-}clA$,*

(iv) *intersection of an arbitrary family of β-θ-closed sets in X is β-θ-closed in X,*

(v) *A is β-θ-open if and only if for each $x \in A$, there exists $V \in \beta R(X, x)$ such that $x \in V \subset A$,*

(vi) *If* $A \in \beta O(X)$ *then* $\beta clA = \beta\text{-}\theta\text{-}clA$,

(vii) *If* $A \in \beta R(X)$ *then* A *is* β-θ-*closed.*

Remark 2.6 [30] Noiri has shown that β-regular $\Rightarrow$ β-θ-open $\Rightarrow$ β-open. But the converses are not necessarily true.

A space X is called β-closed [7] if every β-open cover has a finite subfamily whose β-closures cover X. Equivalently, X is β-closed [7] if every β-regular (resp. β-θ-open) cover has a finite subcover.

3. β-θ-cluster sets and β-δ-cluster sets

Definition 3.1 Let $\psi : (X,\tau) \rightarrow (Y,\tau')$ be a function. Then β-θ-cluster (resp. β-δ-cluster) set of ψ at $x \in X$, denoted by $\beta^{\tau}_{\theta}(\psi; x)$ (resp. $\beta^{\tau}_{\delta}(\psi; x)$), is defined to be the set $\bigcap\{\theta\text{-}cl\psi(\beta clU) : U \in \beta O(X,x)\}$ (resp. $\bigcap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X,x)\}$).

When no topology is mentioned on X then we write $\beta_{\theta}(\psi; x)$ (resp. $\beta_{\delta}(\psi; x)$) instead of $\beta^{\tau}_{\theta}(\psi; x)$ (resp. $\beta^{\tau}_{\delta}(\psi; x)$).

Theorem 3.2 *The following are equivalent for a function* $\psi : X \rightarrow Y$*:*

(a) $y \in \beta_{\delta}(\psi; x)$,

(b) *there is a grill* $\bigwedge$ *on* X *such that* $\bigwedge$ β-θ-*converges to* x *and* $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$,

(c) *the filter base* $\{\psi^{-1}(intclV) : V \in \tau(y)\}$ β-θ-*adheres at* x.

Proof. (a)$\Rightarrow$(c). Let $y \in \beta_{\delta}(\psi; x)$. Then $intclV \cap \psi(\beta clU) \neq \emptyset$ i.e. $\beta clU \cap \psi^{-1}(intclV) \neq \emptyset$ for each $V \in \tau(y)$ and for each $U \in \beta O(X,x)$. Therefore the filter base $\{\psi^{-1}(intclV) : V \in \tau(y)\}$ β-θ-adheres at x.
(c)$\Rightarrow$(b). Let $\bigwedge = \{A \subset X : A \cap F \neq \emptyset, \forall F \in \mathcal{F} = \{\psi^{-1}(intclW) : W \in \tau(y)\}\}$. We shall first show that $\bigwedge$ is a grill on X. It is clear that $\bigwedge \neq \emptyset$ with $\emptyset \notin \bigwedge$ and $A_2 \in \bigwedge$ whenever $A_1 \in \bigwedge$ with $A_1 \subset A_2$. If $A_1 \cup A_2 \in \bigwedge$ i.e. if $(A_1 \cup A_2) \cap F \neq \emptyset$, for all $F \in \mathcal{F}$ then either $A_1 \in \bigwedge$ or $A_2 \in \bigwedge$. Suppose $A_1 \notin \bigwedge$ and $A_2 \notin \bigwedge$. Then there exist an $F_1 \in \mathcal{F}$ such that $A_1 \cap F_1 = \phi$ and an $F_2 \in \mathcal{F}$ such that $A_2 \cap F_2 = \phi$. Since $\mathcal{F}$ is a filter base on X, there exists an $F_3 \in \mathcal{F}$ such that $F_3 \subset F_1 \cap F_2$ and hence $(A_1 \cup A_2) \cap F_3 = \emptyset$ — a contradiction. So, $\bigwedge$ is a grill on X. Let $W \in \tau(y)$ and $V \in \beta O(X,x)$. Since $\mathcal{F}$ β-θ-adheres at x (by hypothesis (c)), $\beta clV \cap \psi^{-1}(intclW) \neq \emptyset$. Hence $\beta clV \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. The definition of $\bigwedge$ shows that $\beta clV \in \bigwedge$ for all $V \in \beta O(X,x)$. Hence the grill $\bigwedge$ β-θ-converges to x. From the definition of $\bigwedge$, it is clear that $intclW \cap \psi(A) \neq \emptyset$ for all $A \in \bigwedge$. So $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$.
(b)$\Rightarrow$(a). Let $\bigwedge$ be a grill on X that β-θ-converges to x and for that grill $y \in \cap\{\delta\text{-}cl\psi(A) : A \in \bigwedge\}$. Hence $\beta clU \in \bigwedge$, for each $U \in \beta O(X,x)$ (as $\bigwedge$ β-θ-converges to x) and thus $y \in \cap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X,x)\} = \beta_{\delta}(\psi; x)$. ∎

Theorem 3.3 *The following are equivalent for a function* $\psi : X \rightarrow Y$*:*

(a) $y \in \beta_\theta(\psi; x)$,

(b) *there is a grill* $\bigwedge$ *on* X *such that* $\bigwedge$ β-θ*-converges to* x *and* $y \in \bigcap\{\theta\text{-}cl\psi(A) : A \in \bigwedge\}$,

(c) *the filter base* $\{\psi^{-1}(clV) : V \in \tau(y)\}$ β-θ*-adheres at* x.

Proof. The proof is similar to the proof of Theorem 3.2. ∎

It is clear from the definitions that for a function $\psi : X \rightarrow Y$, $\beta_\delta(\psi; x) \subseteq \beta_\theta(\psi; x)$, for each point $x \in X$. However, we give an example of a function where the later inclusion is proper.

Example 3.4 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Clearly, $\beta R(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{b, c\}, \{a, c\}\}$. Consider the identity function $\psi : (X, \tau) \rightarrow (X, \tau)$. Then $\beta_\theta(\psi; c) = X$ whereas $\beta_\delta(\psi; c) = \{c\}$. So, $\beta_\delta(\psi; c) \underset{\neq}{\subset} \beta_\theta(\psi; c)$.

The following theorem shows when such cluster sets are equal.

Theorem 3.5 *Let* $\psi : X \rightarrow Y$ *be a function. Then* $\beta_\theta(\psi; x) = \beta_\delta(\psi; x)$*, when* Y *is regular.*

Proof. Since for a regular space Y, δ-$cl(A) = \theta$-$cl(A)$ for each $A \subset Y$, the proof therefore quite obvious. ∎

Definition 3.6 A function $\psi : X \rightarrow Y$ is strongly β-irresolute [30] if for each $x \in X$ and each $V \in \beta O(Y, \psi(x))$, there exists an $U \in \beta O(X, x)$ such that $\psi(\beta clU) \subset V$.

Theorem 3.7 *A space* Y *is weakly Hausdorff if and only if for any space* X*, and any strongly* β*-irresolute surjective function* $\psi : X \rightarrow Y$*,* $\beta_\delta(\psi; x)$ *is degenerate for each* $x \in X$*.*

Proof. *Sufficiency part.* Let for a space X, $\psi : X \rightarrow Y$ be a strongly β-irresolute surjective function. So for points y, z in Y with $y \neq z$ $\exists x$ and $x_0 \in X$ such that $\psi(x) = y$ and $\psi(x_0) = z$. Because of degeneracy of $\beta_\delta(\psi; x)$ and $\beta_\delta(\psi; x_0)$, we have $\beta_\delta(\psi; x) = \{\psi(x)\} = \{y\}$ and $\beta_\delta(\psi; x_0) = \{\psi(x_0)\} = \{z\}$. So there exist $V \in \tau(y)$ and $W \in \beta O(X, x_0)$ such that $\psi(\beta clW) \cap intclV = \emptyset$ and hence $\psi(\beta clW) \subset Y - intclV{=}U$ (say), which is of course a regular closed set of Y. Clearly $z = \psi(x_0) \in U$ but $y \notin U$. Therefore Y is weakly Hausdorff.

Necessity part. Let for any space X, $\psi : X \rightarrow Y$ be a surjective strongly β-irresolute function. Hence for each $x \in X$ and each $V \in \beta O(Y, \psi(x))$, there exists an $W \in \beta O(X, x)$ such that $\psi(\beta clW) \subset V$. From the definition of β-δ-cluster set of ψ at x, it is clear that $\beta_\delta(\psi; x){=}\bigcap\{\delta\text{-}cl\psi(\beta clU) : U \in \beta O(X, x)\} \subseteq \bigcap\{\delta\text{-}clV) : V \in \beta O(Y, \psi(x))\}$. Since Y is weakly Hausdorff for each $y \neq \psi(x)$, there exists a regular closed set U containing $\psi(x)$ such that $y \notin U$. As ψ is strongly β-irresolute and every regular closed set is β-open, there exists a $W \in \beta O(X, x)$ such that $\psi(\beta clW) \subset U$. Since $Y - U = intcl(Y - U) \cap U = \emptyset$, $intcl(Y - U) \cap \psi(\beta clW) = \emptyset$. So, $y \notin \delta\text{-}cl\psi(\beta clW)$ and hence $y \notin \beta_\delta(\psi; x)$. Thus $\beta_\delta(\psi; x)=\{\psi(x)\}$ for each $x \in X$. ∎

A function $\psi : X \to Y$ is called strongly θ-β-continuous [30] if each point $x \in X$ and each open set U containing $\psi(x)$, there exists a $V \in \beta O(X, x)$ such that $\psi(\beta clV) \subset U$.

Theorem 3.8 *A space Y is Hausdorff if and only if for any space X and for any strongly θ-β-continuous surjective function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Let Y be a Hausdorff space and for any space X, $\psi : X \to Y$ is strongly θ-β-continuous surjection. Let $y \in Y$ with $y \neq \psi(x)$. Since $\psi : X \to Y$ is strongly θ-β-continuous, for each open set U containing $\psi(x)$, there is a $V \in \beta O(X, x)$ such that $\psi(\beta clV) \subset U$. Now $\beta_\theta(\psi; x)=\cap\{\theta\text{-}cl\psi(\beta clV) : V \in \beta O(X, x)\} \subseteq \cap\{\theta\text{-}clU : U \in \tau(\psi(x))\} = \cap\{clU : U \in \tau(\psi(x))\}$. Since Y is Hausdorff, there exist disjoint open sets W_1 , W_2 with $y \in W_1$ and $\psi(x) \in W_2$. Hence $W_1 \cap clW_2=\emptyset$. As $y \notin clW_2$, then $y \notin \beta_\theta(\psi; x)$. Thus $\beta_\theta(\psi; x)=\{\psi(x)\}$.

Conversely, let for a space X and for any strongly θ-β-continuous surjective function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$. So, for any two points y, z in Y with $y \neq z$ $\exists x$ and $x_0 \in X$ such that $y = \psi(x)$ and $z = \psi(x_0) \notin \beta_\theta(\psi; x)$. Hence there exist an $U \in \tau(z)$ and a $W \in \beta O(X, x)$ such that $clU \cap \psi(\beta clW)=\phi$ i.e. $\psi(\beta clW) \subset Y - clU$. Since $Y - clU \in \tau(y)$ and $U \in \tau(z)$ then Y is Hausdorff. ∎

Theorem 3.9 *Let Y be an Urysohn space. Then for some space X and for some function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Suppose Y is Urysohn. Suppose also that for each space X and every function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is not degenerate at some point $x \in X$. So for the identity function $f : Y \to Y$, there exist point, say x of Y and a point $y \in Y$ with $y \neq x$ $(= f(x)) \in \beta_\theta(f; x)$. Then for each $W \in \beta O(X, x)$ and each $V \in \tau(y)$, $\beta clW \cap clV \neq \phi$. Hence, in particular, $\beta clW \cap clV \neq \phi$ for each $W \in \tau(x)$ and each $V \in \tau(y)$. But for an open set W, one can cheek that $\beta clW = intclintW = intclW$. So, $intclW \cap clV \neq \emptyset$ and hence $clW \cap clV \neq \emptyset$ for each $W \in \tau(x)$ and each $V \in \tau(y)$. This contradicts that Y is Urysohn. ∎

Theorem 3.10 *A regular topological space Y is Urysohn if for some space X and some surjective function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Let for some space X and for some surjection $\psi : X \to Y$, where Y is regular, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$. So, for $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $\exists$ $x_1, x_2 \in X$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. The degeneracy of $\beta_\theta(\psi; x)$ at each $x \in X$ ensures that $y_1 = \psi(x_1) \notin \beta_\theta(\psi; x_2)$. So, there exists $V \in \tau(y_1)$ and

$W \in \beta O(X,x)$ such that $clV \cap \psi(\beta clW)$=$\emptyset$. Clearly $Y - clV \in \tau(y_2)$. Since Y is regular, there exists an open set $U \in \tau(y_2)$ such that $y_2 \in U \subset clU \subset Y - clV$. So we have $V \in \tau(y_1)$ and $U \in \tau(y_2)$ with $clV \cap clU = \emptyset$. Thus Y is Urysohn. ■

Corollary 3.11 *A regular topological space Y is Urysohn if and only if for some space X and some function $\psi : X \to Y$, $\beta_\theta(\psi; x)$ is degenerate for each $x \in X$.*

In the above discussion, we have observed that the degeneracy of $\beta_\delta(\psi; x)$ and $\beta_\theta(\psi; x)$ at each point x of the domain space X characterize certain separation axioms of the codomain space Y. We now investigate some other situations where separation axioms like almost regularity, Hausdorffness on the domain space ensure the degeneracy of such cluster sets of certain functions.

Definition 3.12 A function $\psi : X \to Y$ is called a θ-closed [13] (resp. δ-closed [29]) function if image of each θ-closed (resp. δ-closed) set in X is θ-closed (resp. δ-closed) in Y.

A δ-closed function $\psi : X \to Y$ is called δ-perfect [29] if each fiber is an NC-set in X.

Theorem 3.13 *Let X be a Hausdorff space. Then for any δ-perfect function $\psi : X \to Y$ (where Y is any space), $\beta_\delta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. Since for any subset S of X, $\beta clS \subseteq \delta\text{-}clS$, then we have $\beta_\delta(\psi; x) = \cap\{\delta - cl\psi(\beta clW) : W \in \beta O(X,x)\} \subseteq \cap\{\delta - cl\psi(\delta - clW) : W \in \beta O(X,x)\}$. Since ψ is δ-perfect and hence is a δ-closed function, then

$$\beta_\delta(\psi; x) \subseteq \cap\{\psi(\delta - clW) : W \in \beta O(X,x)\}.$$

Again since ψ is δ-perfect, $\psi^{-1}(y)$ is an NC-set for each $y \in Y$. It is obvious that every NC-set in a Hausdorff space is a θ-closed set. Let $y \neq \psi(x)$. As $\psi^{-1}(y)$ is θ-closed, there exists an open set $V \in \tau(x)$ such that $\psi^{-1}(y) \cap clV = \emptyset$. Hence $y \notin \psi(clV) = \psi(\delta\text{-}clV)$ (as V is open). Since $\tau(x) \subset \beta O(X,x)$, then from above deduction, $y \notin \beta_\delta(\psi; x)$. Therefore $\beta_\delta(\psi; x) = \{\psi(x)\}$ for each $x \in X$. ■

Definition 3.14 A subset S of space (X,τ) is called α-paracompact [23] (resp. α-nearly paracompact [23]) if every cover of S by open (resp. regular open) sets of (X,τ) has an open X-locally finite refinement.

Theorem 3.15 *Let X is a Hausdorff space. Then for any δ-closed function $\psi : X \to Y$ (where Y is any space), if $\psi^{-1}(y)$ is α-nearly paracompact for each $y \in Y$, then $\beta_\delta(\psi; x)$ is degenerate for each $x \in X$.*

Proof. As every α-nearly paracompact set in a Hausdorff space is θ-closed [15], the proof is quite similar to the proof of the Theorem 3.13. ■

Theorem 3.16 *Let X is a Hausdorff paracompact space. Then for any δ-closed function $\psi : X \to Y$ (where Y is any space), if for each $y \in Y$ $\psi^{-1}(y)$ is any of the following:*

(a) *α-paracompact,*
(b) *α-nearly paracompact,*
(c) *δ-closed,*
(d) *θ-closed,*
(e) *closed,*
(f) *g-closed.*

Then, $\beta_\delta(\psi;x)$ is degenerate for each $x \in X$.

Proof. Dontchev and Noiri (Corollary 3.7 [15]), have proved that for a subset A of Hausdorff paracompact space the following are equivalent: (a) A is α-paracompact, (b) A is α-nearly paracompact, (c) A is δ-closed, (d) A is θ-closed, (e) A is closed, (f) A is g-closed. So the theorem follows from Theorem 3.15. ∎

Definition 3.17 A space (X,τ) is called almost regular [34] if for every regular closed set S in X and for each $x \in S$, there exist disjoint open sets U and V such that $x \in U$ and $S \subset V$.

Theorem 3.18 *Let $\psi : X \to Y$ be θ-closed map from a almost regular space X into a space Y. If $\psi^{-1}(y)$ is θ-closed in X for each $y \in Y$, then $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$.*

Proof. Since for each subset A of X, $\beta clA \subset \theta\text{-}clA$ and since in a almost regular space X, $\theta\text{-}clA$ is always θ-closed the proof is thus quite similar to the proof of Theorem 3.13 and is therefore omitted. ∎

Theorem 3.19 *Let X be an almost regular Hausdorff space. If $\psi : X \to Y$ (where Y is any space) is an injective θ-closed function then $\beta_\theta(\psi;x)$ is degenerate for each $x \in X$.*

Proof. Since X be an almost regular then from the argument given in Theorem 3.18, we get $\beta_\theta(\psi;x) \subseteq \{\psi(\theta\text{-}clV) : V \in \beta O(X,x)\}$. Now as X is being Hausdorff, for any point x_1 in X with $x \neq x_1$, there is an open set $W \in \tau(x) \subseteq \beta O(X,x)$ such that $x_1 \notin \theta\text{-}clW$. Since ψ is injective, so $\psi(x_1) \notin \psi(\theta\text{-}clW)$. Therefore $\beta_\theta(\psi;x) = \{\psi(x)\}$. ∎

For a σ-ideal $\mathcal{I}$ on X, the two topologies τ_1 and τ_2 on X are called equivalent modulo $\mathcal{I}$, denoted by $\tau_1 \equiv \tau_2(mod\mathcal{I})$ if for any subset S of X, the difference set $(\tau_1)\beta\text{-}\theta\text{-}clS - (\tau_2)\beta\text{-}\theta\text{-}clS \in \mathcal{I}$, where $(\tau_i)\beta\text{-}\theta\text{-}clS$ is the β-θ-closure of S in the space (X,τ_i) for $i = 1,2$.

Theorem 3.20 *Let τ_1 and τ_2 be two topologies on X such that $\tau_1 \equiv \tau_2(mod\mathcal{I})$, where $\mathcal{I}$ is a σ-ideal of subsets of X and also let Y be a second countable space. Then for any function $\psi : X \to Y$, there is an $I \in \mathcal{I}$ such that $\beta_\delta^{\tau_1}(\psi;x) = \beta_\delta^{\tau_2}(\psi;x)$, for each $x \in X - I$.*

Proof. Consider the sets

$$L = \{x \in X : \beta_\delta^{\tau_1}(\psi;x) \nsubseteq \beta_\delta^{\tau_2}(\psi;x)\} \text{ and } R = \{x \in X : \beta_\delta^{\tau_2}(\psi;x) \nsubseteq \beta_\delta^{\tau_1}(\psi;x)\}.$$

Since Y is second countable it has a countable base, (say) $\mathcal{B} = \{V_n\, n \geq 1\}$. If we take $L_n = \{x \in X : x \in (\tau_1)\beta-\theta-cl\psi^{-1}(\beta clV_n)$ but $x \notin (\tau_2)\beta-\theta-cl\psi^{-1}(\beta clV_n)\}$, for each $n = 1, 2, ...$, then each $L_n \in \mathcal{I}$ and hence $\cup\{L_n : n \geq 1\} \in \mathcal{I}$ (as $\tau_1 \equiv \tau_2(mod\mathcal{I})$ and $\mathcal{I}$ is a σ-ideal). We claim that $L \subseteq \cup\{L_n : n \geq 1\}$. Indeed, let $x \in L$ then their exists an $y_0 \in \beta_\delta^{\tau_1}(\psi; x)$ but $y_0 \notin \beta_\delta^{\tau_2}(\psi; x)$. Theorem 3.2 asserts that the filter base $\bigwedge = \{\psi^{-1}(\beta clV) : V \in \tau(y_0)\}$ β-θ-adheres at x in (X, τ_1) but not in (X, τ_2). So there exists an open set $U \in \tau(y_0)$ such that $x \notin (\tau_2)\beta$-θ-$cl\psi^{-1}(\beta clU)$ and hence $x \notin (\tau_2)\beta$-θ-$cl\psi^{-1}(\beta clV_n)$ for some $V_n \in \mathcal{B}$ for which $y_0 \in V_n \subset U$. Obviously, $x \in (\tau_1)\beta$-θ-$cl\psi^{-1}(\beta clV_n)$. Hence $x \in L_n$. So $L \subseteq \cup\{L_n : n \geq 1\}$ and since $\mathcal{I}$ is a σ-ideal, $L \in \mathcal{I}$. Similarly $R \in \mathcal{I}$. Thus the set $\{x \in X : \beta_\delta^{\tau_1}(\psi; x) \neq \beta_\delta^{\tau_2}(\psi; x)\} = L \cup R \in \mathcal{I}$. Hence the proof is complete. ∎

Theorem 3.21 *Let τ_1 and τ_2 be two topologies on X with $\tau_1 \equiv \tau_2(mod\mathcal{I})$, where $\mathcal{I}$ is a σ-ideal of subsets of X and also let Y be a second countable space. Then for any function $\psi : X \to Y$, there is an $I \in \mathcal{I}$ such that $\beta_\theta^{\tau_1}(\psi; x) = \beta_\theta^{\tau_2}(\psi; x)$, for each $x \in X - I$.*

Proof. The proof is similar to the above theorem and is thus omitted. ∎

We now establish a theorem which asserts that under certain conditions the equivalence of $\beta_\delta(\psi; x)$, $\beta_\theta(\psi; x)$ and some other cluster sets which are already found in literature like P-cluster set $P(\psi; x)$ [27], S-cluster set $S(\psi; x)$ [26], cluster set $C(\psi; x)$ [40]. For this, we need the following definition.

Definition 3.22 A space (X, τ) is said to be submaximal [10] if every dense subset of X is open. It is called extremally disconnected [28] if closure of each open set is open in X.

Theorem 3.23 *Let X be a submaximal extremally disconnected space and Y be a regular space then for a function $\psi : X \to Y$ the following sets are all equal:*

(a) $\beta_\delta(\psi; x)$,
(b) $\beta_\theta(\psi; x)$,
(c) $C(\psi; x)$,
(d) $S(\psi; x)$,
(e) $P(\psi; x)$.

Proof. Since the space (X, τ) is a submaximal extremally disconnected then $\tau = \tau_\alpha = SO(X) = PO(X) = \beta O(X)$ [8]. Hence the proof follows. ∎

4. β-θ-cluster set and β-δ-cluster set for multifunctions and characterizations of β-closed spaces

Definition 4.1 Let $F : X \to Y$ be a multifunction and $x \in X$. Then β-θ-cluster (resp. β-δ-cluster) set of X at $x \in X$, is denoted by $\beta_\theta(F; x)$ (resp. $\beta_\delta(F; x)$), is defined as the set $\bigcap\{\theta\text{-}clF(\beta clU) : U \in \beta O(X, x)\}$ (resp. $\bigcap\{\delta\text{-}clF(\beta clU) : U \in \beta O(X, x)\}$).

For a multifunction $F : X \to Y$ and a subset S of X, the notation $\beta_\theta(F; S)$ (resp. $\beta_\delta(F; S)$) stands for the set $\bigcup_{x\in S} \beta_\theta(F; x)$ (resp. $\bigcup_{x\in S} \beta_\delta(F; x)$).

Definition 4.2 A multifunction $F : X \to Y$ is said to have a β-θ-closed (resp. β-δ-closed) graph if for each $(x, y) \notin G(F)$, there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $(\beta clU \times clV) \cap G(F) = \emptyset$ (resp. $(\beta clU \times intclV) \cap G(F) = \emptyset$).

Definition 4.3 For a subset $S \subset X \times Y$, the (2)β-θ-closure of S, denoted by (2)β-θ-$cl(S)$ is defined as the set $\{(x, y) \in X \times Y$: for all $V \in \beta O(X, x)$ and for all $W \in \tau(y)$, $(\beta clV \times clW) \cap S \neq \emptyset\}$. If $S =$ (2)β-θ-$cl(S)$, then S is called (2)β-θ-closed.

Theorem 4.4 *Let $F : X \to Y$ be a multifunction. Then*

(a) $\beta_\theta(F; x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$ *for each* $x \in X$,

(b) $\beta_\delta(F; x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\delta\text{-}clG(F))$ *for each* $x \in X$.

Proof. (a) Let $x \in X$. Then $y \in \beta_\delta(F; x)$ if and only if for each $U \in \beta O(X, x)$ and each $V \in \tau(y)$, $clV \cap F(\beta clU) \neq \emptyset$ if and only if $(\beta clU \times clV) \cap G(F) \neq \emptyset$ if and only if $(x, y) \in (2)\beta\text{-}\theta\text{-}clG(F)$ if and only if $y \in \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$.
(b) The proof is quite similar to (a) and is thus omitted. ∎

Theorem 4.5 *Let $F : X \to Y$ be a multifunction. Then following statements are equivalent:*

(a) $\beta_\theta(F; x) = F(x)$ *for each* $x \in X$,

(b) *F has a β-θ-closed graph,*

(c) $F(x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$ *for each* $x \in X$.

Proof. (a)⇒(b). Let $(x, y) \notin G(F)$. Then $y \notin F(x)$. Hence by (a), $y \notin \beta_\theta(F; x)$. So there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $clV \cap F(\beta clU) = \emptyset$. Thus $(\beta clU \times clV) \cap G(F) = \emptyset$. So F has a β-θ-closed graph.
(b)⇒(c). Let $x \in X$ and $y \notin F(x)$. Then $(x, y) \notin G(F)$. Since F has a β-θ-closed graph, there exist $U \in \beta O(X, x)$ and $V \in \tau(y)$ such that $(\beta clU \times clV) \cap G(F) = \emptyset$ and hence $(x, y) \notin (2)\beta\text{-}\theta\text{-}clG(F)$. Thus $y \notin \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\theta\text{-}clG(F))$. The reverse inclusion is also similar.
(c)⇒(a). Follows from Theorem 4.4 (a). ∎

Theorem 4.6 *For a multifunction $F : X \to Y$ the following statements are equivalent:*

(a) $\beta_\delta(F; x) = F(x)$ *for each* $x \in X$,

(b) *F has a β-δ-closed graph,*

(c) $F(x) = \Pi_y((\{x\} \times Y) \cap (2)\beta\text{-}\delta\text{-}clG(F))$ *for each* $x \in X$.

Proof. The proof is quite similar to Theorem 4.5. ∎

Definition 4.7 A multifunction $F : X \to Y$ is said to be upper strongly θ-β-continuous if for each $x \in X$ and each $V \in \tau(F(x))$, there is an $U \in \beta O(X, x)$ such that $F(\beta clU) \subset V$.

Theorem 4.8 *If $F : X \to Y$ is upper strongly θ-β-continuous multifunction, then for each $x \in X$, $\beta_\theta(F; x) = \theta\text{-}clF(x)$.*

Proof. From the definition of $\beta_\theta(F; x)$, it is clear that $\theta\text{-}clF(x) \subset \beta_\theta(F; x)$. Since F is a upper strongly θ-β-continuous, for each $V \in \tau(F(x))$, there is an $U \in \beta O(X, x)$ such that $F(\beta clU) \subset V$. Hence the filter subbase $F(\beta R(X, x))$ is stronger than the filter subbase $\tau(F(x))$ in Y. So $\beta_\theta(F; x) \subset ad_\theta\tau(F(x)) = ad\tau(F(x)) = \theta\text{-}clF(x)$ (since for a subset K of a space X, $\theta\text{-}clK = ad\tau(K)$ [22]). Therefore $\beta_\theta(F; x) = \theta\text{-}clF(x)$. ∎

Theorem 4.9 *An upper strongly θ-β-continuous multifunction has a β-θ-closed graph if and only if it has θ-closed point images.*

Proof. The proof follows from Theorem 4.5 and Theorem 4.8. ∎

Corollary 4.10 *A strongly θ-β-continuous [30] surjection $\psi : X \to Y$ has a β-θ-closed graph if and only if Y is Hausdorff.*

Proof. Since a space Y is Hausdorff if and only if its points are θ-closed, the proof follows from Theorem 4.9. ∎

We say a multifunction $F : X \to Y$ has a θ-closed graph (resp.δ-closed graph) if $G(F)$ is a θ-closed (resp.δ-closed) subset of $X \times Y$.

Theorem 4.11 *If a multifunction $F : X \to Y$ has*

(i) *θ-closed graph then $\beta_\theta(F; x) = F(x)$ for each $x \in X$,*

(ii) *δ-closed graph then $\beta_\delta(F; x) = F(x)$, for each $x \in X$.*

Proof. We proof this theorem for the multifunction having θ-closed graph only. For the case of multifunction having δ-closed graph, the proof is similar. Let $x \in X$ and $y \in \beta_\theta(F; x)$. Then for each $V \in \beta O(X, x)$ and each $U \in \tau(y)$, $F(\beta clV) \cap clU \neq \emptyset$ i.e $F^-(clU) \cap \beta clV \neq \emptyset$. Thus for any basic open set $W_1 \times W_2$ in $X \times Y$ containing (x, y), $F^-(clW_1) \cap F^-(clW_2) \neq \emptyset$ (as $\tau(x) \subset \beta O(X, x)$) and hence $cl(W_1 \times W_2) = (clW_1 \times clW_2) \cap G(F) \neq \emptyset$ (as $\beta clW_1 \subset clW_1$). So $(x, y) \in \theta\text{-}clG(F) = G(F)$. Hence $y \in F(x)$. Thus $\beta_\theta(F; x) \subseteq F(x)$. Also it is obvious that $F(x) \subseteq \beta_\theta(F; x)$. So $\beta_\delta(F; x) = F(x)$, for each $x \in X$. ∎

Lemma 4.12 *Let X and Y be topological spaces, $M \subseteq X$, $N \subseteq Y$. Then*

(i) *$\beta cl(M \times N) \subseteq \beta clM \times \beta clN$,*

(ii) *If $M \in \beta O(X, x)$ and $N \in \beta O(Y, y)$ then $M \times N \in \beta O(X \times Y, (x, y))$.*

Proof. (i) $\beta cl(M \times N) = (M \times N) \cup intclint(M \times N) = (M \times N) \cup (intclintM \times intclintN) \subseteq (M \cup intclintM) \times (N \cup intclintN) = \beta clM \times \beta clN$.
(ii) $M \times N \subseteq (clintclM) \times (clintclN) = clintcl(M \times N)$. ∎

Theorem 4.13 *If a multifunction $F : X \to Y$ satisfies any one of the following:*

(a) $\beta_\theta(F; x) = F(x)$, *for each* $x \in X$,

(b) $\beta_\delta(F; x) = F(x)$, *for each* $x \in X$.

Then, the graph $G(F)$ is sp-θ-closed [30].

Proof. (a) Let $F : X \to Y$ be the multifunction satisfies (a) and let $(x, y) \notin G(F)$. Then $y \notin F(x) = \beta_\theta(F; x)$ and so there exist $U \in \beta O(X, x)$ and $V \in \tau(y) \subseteq \beta O(Y, y)$ such that $clV \cap F(\beta clU) = \emptyset$ and hence $(\beta clU \times \beta clV) \cap G(F) = \emptyset$ (as $\beta clU \subset clV$). But from the lemma 4.12, we have $\beta cl(U \times V) \cap G(F) = \emptyset$, where $U \times V \in \beta O(X \times Y, (x, y))$. So $G(F)$ is *sp*-θ-closed.
(b) Since for a subset S of Y, $\beta clintS = intclintS$ [6], then for any open set V of Y $\beta clV = intclV$. Because of this fact, the proof is now quite similar to the proof as for the case (a). ∎

Theorem 4.14 *A subset S of a topological space X is β-closed relative to X (i.e. β-set) if and only if for every filter base Ω on X with $F \cap K \neq \emptyset$ for each $F \in \Omega$ and each $K \in \beta O(X, S)$, $S \cap \beta$-θ-$ad\Omega \neq \emptyset$.*

Proof. Let S be not β-closed relative to X. Then there exists a cover $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ by β-open sets of X such that $S \not\subset \cup\{\beta clV_\alpha : \alpha \in I_0\}$ for any finite subset I_0 of I. Clearly the family $\Omega = \{S - \cup_{\alpha \in I_0} \beta clV_\alpha : I_0$ is a finite subset of $I\}$ is a filterbase on X. As $\emptyset \neq F \subseteq S$ and $S \subseteq K$ for each $F \in \Omega$ and $K \in \beta O(X, S)$ then $F \cap K \neq \emptyset$ for each $F \in \Omega$ and $K \in \beta O(X, S)$. Clearly $S \cap \beta$-θ-$ad\Omega = \emptyset$. In fact if $x \in S$ then there exists $V_\alpha \in \mathcal{V}$ such that $x \in V_\alpha$ and $X - \beta clV_\alpha \in \Omega$ satisfying $\beta clV_\alpha \cap (X - \beta clV_\alpha) = \emptyset$, a contradiction. So S is β-closed relative to X.

Conversely, if possible let $S \cap \beta$-θ-$ad\Omega = \emptyset$, when S is β-closed relative to X and Ω satisfies condition of the hypothesis. Then for each $x \in S$, there exists a $V_x \in \beta O(X, x)$ and an $F_x \in \Omega$ such that $\beta clV_x \cap F_x = \emptyset$. Since S is β-closed relative to X, there exist $x_1, x_2, ..., x_n \in S$ such that $S \subset \cup_{i=1}^n \beta clV_{x_i}$. Since Ω is a filterbase, there exists an $F \in \Omega$ such that $F \subseteq \cup_{i=1}^n \beta clF_{x_i}$. Clearly $F \cap (\cup_{i=1}^n \beta clV_{x_i}) = \emptyset$ and $\cup_{i=1}^n \beta clV_{x_i} \in \beta O(X, S)$. This contradicts the hypothesis which Ω satisfies. ∎

Lemma 4.15 *If S is a β-θ-closed subset of a β-closed space X then S is β-closed relative to X.*

Proof. The proof is quite clear. ∎

Theorem 4.16 *The following statements are equivalent for a space X:*

(a) *X is β-closed,*

(b) *For a multifunction $F : X \to Y$, where Y is any topological space,* $\cap\{\delta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\delta(F, S)$, *for each β-δ-closed set S of X,*

(c) *For each multifunction $F : X \to Y$, where Y is any topological space,* $\cap\{\beta\text{-}\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$, *for each β-θ-closed set S of X.*

Proof. (a)⇒(b). Since X is β-closed and S is β-θ-closed, then by Lemma 4.15, S is β-closed and X. If $y \in \cap\{\delta\text{-}clF(V) : V \in \beta O(X, S)\}$ then for each $U \in \tau(y)$, $F(V) \cap intclU \neq \emptyset$ i.e. $V \cap F^-(intclV) \neq \emptyset$. Since S is β-closed relative to X and the filter base $\Omega = \{F^-(intclU) : U \in \tau(y)\}$ satisfies the condition of Theorem 4.14, then $S \cap \beta\text{-}\theta\text{-}ad\Omega \neq \emptyset$. Let $x \in S \cap \beta\text{-}\theta\text{-}ad\Omega$. Then for each $W \in \beta O(X, x)$ and each $U \in \tau(y)$, $F^-(intclU) \cap \beta clW \neq \emptyset$ i.e. $intclU \cap F(\beta clW) \neq \emptyset$; which shows that $y \in \delta\text{-}clF(\beta clW)$, for each $W \in \beta O(X, x)$. Thus $y \in \beta_\delta(F, x) \subseteq \beta_\delta(F, S)$.
(b)⇒(c). Since we know that for a subset A of X, $\beta clintA = intclintA$ [6], then for an open U, $\beta clU = intclU$. Hence from the definition of β-θ-closure of a subset A, we have $\beta\text{-}\theta\text{-}clA \subseteq \delta\text{-}clA$. So (b)⇒(c) follows immediately.
(c)⇒(a). Let Ω be a filter base on X. Let $z \notin X$. Now consider $Y = X \cup \{z\}$ and $\tau_Y = \{U \subseteq Y$: either $z \notin U$ or there exists $F \in \Omega$ such that $F \subseteq U$ when $z \in U\}$. Then τ_Y is a topology on Y [22]. So for the identity map $\psi : X \to Y$, we have from hypothesis (c) that $\beta_\delta(\psi, X) \supseteq \cap\{\beta\text{-}\theta\text{-}cl\psi(V) : V \in \beta O(Y, X)\} = \cap\{\beta\text{-}\theta\text{-}clV : V \in \beta O(Y, X)\} = \beta\text{-}\theta\text{-}clX$ (in (Y, τ_Y)). To prove the last equality, it is enough to prove that $z \in \beta\text{-}\theta\text{-}clX$ (in (Y, τ_Y)). Clearly $\{z\}$ is not open in (Y, τ_Y) and also $\{z\}$ is not β-open in (Y, τ_Y). In fact $\{z\} \not\subseteq clintcl\{z\}$ (as $cl\{z\} = \{z\}$ and hence $intcl\{z\} = \emptyset$). So $\beta clU \cap X \neq \emptyset$, for all $U \in \beta O(Y, z)$. Thus $z \in \beta\text{-}\theta\text{-}clX$. Hence $z \in \beta_\delta(\psi, x)$ for some $x \in X$. Let $F \in \Omega$. Then from the construction of τ_Y, it is clear that $Y - (F \cup \{z\})$ as well as $F \cup \{z\}$ are both open in (Y, τ_Y). Also $\beta cl(F \cup \{z\}) = intcl(F \cup \{z\})$ (as for an open set W, $\beta clW = intclW$). So for each $V \in \beta O(X, x)$, and each $F \in \Omega$, we have $\beta clV \cap F = \psi(\beta clV) \cap (F \cup \{z\})$ (as $z \notin \psi(\beta clV)) = \psi(\beta clV) \cap \beta cl(F \cup \{z\})$ (since $\{x_0\}$ is open and hence β-open in (Y, τ_Y) whenever a point x_0 in X satisfies $x_0 \notin F) = \psi(\beta clV) \cap intcl(F \cup \{z\}) \neq \emptyset$ (as $z \in \beta_\delta(\psi, x)$ and $F \cup \{z\} \in \tau_Y$). So $x \in \beta\text{-}\theta\text{-}ad\Omega$ and hence X is β-closed as we know that a space X is β-closed if and only if every filter base on X, β-θ-adheres at a some point in X [7]. ∎

Theorem 4.17 *The following statements are equivalent for a space X:*

(a) *X is β-closed,*

(b) *For a multifunction $F : X \to Y$, where Y is any topological space,* $\cap\{\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$, *for each β-θ-closed set S of X,*

(c) *For each multifunction $F : X \to Y$, where Y is any topological space,* $\cap\{\beta\text{-}\theta\text{-}clF(V) : V \in \beta O(X, S)\} \subseteq \beta_\theta(F, S)$, *for each β-θ-closed set S of X.*

Proof. The proof is quite similar to the proof of the Theorem 4.16 and is thus omitted. ∎

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