Abstract. This paper deals with hyperoperations that derive from binary relations and it studies the hypercompositional structures that are created by them. It is proved that if $\rho$ is a binary relation on a non-void set $H$, then the hypercomposition $xy = \{ z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho \}$ satisfies the associativity or the reproductivity only when it is total. There also appear routines that calculate (with the use of small computing power) the number of non isomorphic hypergroupoids, when the cardinality of $H$ is finite.

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partial hypegroupoid, while it is called hypegroupoid if $ab \neq \emptyset$, for all $a, b \in H$. A hypegroupoid in which the associativity is valid, is called semi–hypergroup, while it is called quasi–hypergroup if only the reproductivity holds.

Several papers dealing with the construction of hypegroupoids and hypergroups appear in the relevant bibliography, since hypergroups are much more varied than groups, e.g. for each prime number $p$, there exists only one group, up to isomorphism, with cardinality $p$, while the number of pairwise non isomorphic hypergroups is very large. For example there exist 3999 non isomorphic hypergroups with 3 elements [12]. Nieminen [8] studied hypergroups associated with graphs and G. G. Massouros studied hypergroups associated with automata [7]. Also Chvalina [1], Rosenberg [9], Corsini [2], De Salvo and Lo Faro [3] studied hypergroupoids and hypergroups defined in terms of binary relations. This paper deals with the hypergroupoids defined by Corsini, it proves that this family of hypergroupoids contains only one semihypergroup and only one quasihypergroup, the total hypergroup and enumerates the hypergroupoids with 2, 3, 4 and 5 elements. The order $n$ of a finite hypergroupoid $H$ is defined to be the number of elements in the set $H$.

Let $H$ be a non empty set and $\rho$ a binary relation on $H$. Corsini introduced in $H$ the hypercomposition.

\[(1.1) \quad x \cdot y = \{ z \in H : (x, z) \in \rho \text{ and } (z, y) \in \rho \} \]

With the above hypercomposition, $(H, \cdot)$ becomes a partial hypergroupoid, while it becomes a hypergroupoid if for each pair of elements $x, y \in H$, there exists $z \in H$ such that $(x, z) \in \rho$ and $(z, y) \in \rho$. Since $\rho^2 = \rho \circ \rho = \{(x, y) \in H^2 : (x, z), (z, y) \in \rho \text{ for some } z \in H\}$, it derives that $(H, \cdot)$ is a hypegroupoid if $\rho^2 = H^2$.

2. The hypercompositional structures defined by $\rho$

Let $H_\rho$ denote the hypercompositional structure defined by (1.1) through the binary relation $\rho$. One can observe that the reproductivity is valid in $H_\rho$ if and only if $(x, y) \in \rho$, for all $x, y \in H_\rho$. Indeed let $x$ be an arbitrary element of $H_\rho$. For the reproductivity to be valid, it must hold: $y \in xH_\rho$, for all $y \in H_\rho$. Hence, for all $x, y \in H_\rho$, the pair $(x, y)$ must belong to $\rho$. Thus:

**Proposition 2.1.** $H_\rho$ is a quasihypergroup, if and only if $(x, y) \in \rho$ for all $x, y \in H_\rho$.

Next, suppose that $H_\rho$ is a hypergroupoid. Then:

**Lemma 2.1.** If $H_\rho$ is a semihypergroup and $(z, z) \notin \rho$ for some $z \in H_\rho$, then $(s, z) \in \rho$ implies that $(z, s) \notin \rho$.

**Proof.** Suppose that $(s, z) \in \rho$ and $(z, s) \in \rho$. Then for $zz$ and $ss$ we have

\[zz = \{ x \in H_\rho : (z, x) \in \rho \text{ and } (x, z) \in \rho \}\]

thus $s \in zz$ and,

\[ss = \{ x \in H_\rho : (s, x) \in \rho \text{ and } (x, s) \in \rho \}\]
thus $z \in ss$. Now $z \in (zz)s$ since $ss \subseteq (zz)s$. But $z \notin z(zs)$, because:

$$z(zs) = \{ x \in H_{\rho} : (z, x) \in \rho \text{ and } (x, s) \in \rho \} = \{ y \in H_{\rho} : (z, y) \in \rho \text{ and } (y, x) \in \rho \}$$

and $(z, z) \notin \rho$. Hence the associativity is not valid, which contradicts the assumption that $H_{\rho}$ is a semihypergroup.

**Corollary 2.1.** If $H_{\rho}$ is a semihypergroup and $\rho$ is not reflexive, then $\rho$ is not symmetric.

**Lemma 2.2.** If $H_{\rho}$ is a semihypergroup, then $\rho$ is reflexive.

**Proof.** Suppose that $(x, x) \notin \rho$, for some $x \in H_{\rho}$. Then, according to Lemma 2.1, for every element $t$ in $H_{\rho}$ such that $(x, t) \in \rho$, it derives that $(t, x) \notin \rho$. But $xx = \{ y \in H_{\rho} : (x, y) \in \rho \text{ and } (y, x) \in \rho \}$. Therefore $xx = \emptyset$, which is absurd, since $H_{\rho}$ is a semihypergroup.

**Lemma 2.3.** If any pair of elements of $H_{\rho}$ does not belong to $\rho$, then $H_{\rho}$ is not a semihypergroup.

**Proof.** According to Lemma 2.2, if $(x, x) \notin \rho$ for some $x \in H_{\rho}$, then $H_{\rho}$ is not a semihypergroup. So, let $t, z$ be two elements of $H_{\rho}$ such that $t \neq z$ and $(t, z) \notin \rho$. Then:

$$t(tz) = t\{ s \in H : (t, s) \in \rho \text{ and } (s, z) \in \rho \} = \{ y \in H : (t, y) \in \rho \text{ and } (y, s) \in \rho \}$$

According to Lemma 2.2, it holds $(t, t) \in \rho$. Also $(t, s) \in \rho$. Therefore $t \in t(tz)$.

On the other hand:

$$(tt)z = \{ r \in H : (t, r) \in \rho \text{ and } (r, t) \in \rho \}z = \{ w \in H : (r, w) \in \rho \text{ and } (w, z) \in \rho \}$$

Thus $(tt)z \subseteq \{ w \in H : (w, z) \in \rho \}$, therefore $t \notin (tt)z$. Hence the associativity is not valid.

From the above series of lemmas, it derives that:

**Proposition 2.2.** $H_{\rho}$ is a semihypergroup if and only if $(x, y) \in \rho$, for all $x, y \in H_{\rho}$.

Now, if $(x, y) \in \rho$ for all $x, y \in H_{\rho}$, then the hypercomposition which is defined through $\rho$ is total, i.e. $xy = H_{\rho}$, for all $x, y \in H_{\rho}$. But if a hypercompositional structure is endowed with the total hypercomposition, then it is a hypergroup. Therefore, from Propositions 2.1 and 2.2, it derives that:

**Theorem 2.1.** The only semihypergroup and the only quasihypergroup defined by the binary relation $\rho$ is the total hypergroup.

### 3. Enumeration of the finite hypergroupoids

Every relation $\rho$ in a finite set $H$ with $\text{card} H = n$, is represented by a Boolean matrix $M_\rho$ and conversely every $n \times n$ Boolean matrix defines in $H$ a binary
relation. Indeed, let $H$ be the set $\{a_1, \cdots, a_n\}$. Then a $n \times n$ Boolean matrix is constructed as follows: the element $(i, j)$ of the matrix is 1, if $(a_i, a_j) \in \rho$ and it is 0 if $(a_i, a_j) \notin \rho$ and vice versa. Hence, in every set with $n$ elements, $2^{n^2}$ partial hypergroupoids can be defined.

Recall that in Boolean algebra it holds: $0 + 1 = 1 + 0 = 1 + 1 = 1$, while $0 + 0 = 0$. Also $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Let $H_\rho$ be the above mentioned partial hypergroupoid, which is defined by a binary relation $\rho$. Then $H_\rho$ is a hypergroupoid if and only if $M_\rho^2 = T$, where $T = (t_{ij})$ with $t_{ij} = 1$ for all $i, j$ [2]. The matrix $M_\rho$ is called good, if $H_\rho$ is a hypergroupoid. Since the element $c_{ij}$ of $M_\rho^2$ is equal to $\sum_{s=1}^n x_{is} y_{sj}$, it derives that matrices having a column or a row consisting only of 0 elements are not good.

Now, from Proposition 2.1 it derives:

**Proposition 3.1.** $H_\rho$ is quasihypergroup if and only if $M_\rho = T$.

Also, Proposition 2.2 gives:

**Proposition 3.2.** $H_\rho$ is semihypergroup if and only if $M_\rho = T$.

Hence, the theorem holds:

**Theorem 3.1.** The only relation $\rho$ that gives a semihypergroup or a quasihypergroup is the one which has $M_\rho = T$, and so $H_\rho$ is the total hypergroup.

Spartalis and Mamaloukas [11] wrote, in Visual Basic code, a 190-lines long program that enumerates the hypergroupoids associated with binary relations of orders 2, 3 and 4. Though, the following few lines of a Mathematica [13] program produces these results through a considerably shorter process. It simply collects in variable $c$ all the Boolean matrices of size $n$ and computes their squares. Boolean minimum entry of these squares is recorded in table $z$. In return, we count the nonzero elements of $z$.

```mathematica
Good[n_] :=
Module[{c, i1, z},
  c = Tuples[Tuples[{0, 1}, n], n];
  z = Table[Min[Flatten[c[[i1]].c[[i1]]]],
    {i1, 1, 2^(n*n)}];
  Return[Count[z, _?Positive]]
]
```

Which gives:

\begin{align*}
\text{In[1]} & := \text{Good[2]} \\
\text{Out[1]} & = 3 \\
\text{In[2]} & := \text{Good[3]} \\
\text{Out[2]} & = 73 \\
\text{In[3]} & := \text{Good[4]} \\
\text{Out[3]} & = 6003 \\
\text{In[4]} & := \text{Good[5]} \\
\text{Out[4]} & = 2318521
\end{align*}
Thus, it is confirmed that there exist 3, 73 and 6003 binary relations that form a hypergroupoid of orders 2, 3 and 4 respectively. It took the above program only a few minutes to count 2318521 hypergroupoids of order 5. For $n = 6$, the function \texttt{Good} fails due to memory restrictions of a small computer. One can proceed with a more slow but reliable package and form one by one the various Boolean matrices and their squares.

\textbf{Remark.} Notice that the above enumeration coincides with the enumeration of square roots of the total Boolean matrix, i.e. the Boolean matrix with all entries equal 1.

\section{3.1. Isomorphisms}

Naturally, the question arises: When two hypergroupoids, are isomorphic?

\textbf{Proposition 3.3.} If in the Boolean matrix $M$, the $i$ and $j$ rows are interchanged and, at the same time, the corresponding $i$ and $j$ columns are interchanged as well, then the deriving new matrix and the initial one, give isomorphic hypergroupoids.

\textbf{Proof.} Suppose that $H = \{a_1, \ldots, a_n\}$ is a finite set and let $(H_{\rho_1}, \bullet_{\rho_1})$ be the hypergroupoid defined by a binary relation $\rho_1$. Let $M_{\rho_1}$ be the Boolean matrix defined by $\rho_1$. Now suppose that the $i$ and $j$ rows and columns are interchanged and let $M_{\rho_2}$ be the new Boolean matrix. Then a new binary relation $\rho_2$ is defined on $H$. Obviously for $\rho_1$ and $\rho_2$ it holds:

\begin{align*}
(a_k, a_i) &\in \rho_1 \iff (a_k, a_j) \in \rho_2 \quad (a_i, a_k) \in \rho_1 \iff (a_j, a_k) \in \rho_2 \quad \text{if} \ k \neq i, j \\
(a_i, a_j) &\in \rho_1 \iff (a_j, a_i) \in \rho_2 \quad (a_j, a_i) \in \rho_1 \iff (a_i, a_j) \in \rho_2 \\
(a_i, a_i) &\in \rho_1 \iff (a_j, a_j) \in \rho_2 \quad (a_j, a_j) \in \rho_1 \iff (a_i, a_i) \in \rho_2
\end{align*}

If $(H_{\rho_2}, \bullet_{\rho_2})$ is the hypercompositional structure defined by $M_{\rho_2}$, then the mapping $\phi : H_{\rho_1} \longrightarrow H_{\rho_2}$ with:

$$\phi(x) = \begin{cases} x & \text{if } x \neq a_i, a_j \\ a_i & \text{if } x = a_j \\ a_j & \text{if } x = a_i \end{cases}$$

is an isomorphism. Obviously $\phi$ is $1 - 1$ and onto. Next we distinguish the cases:

1. $\phi(a_i \bullet_{\rho_1} a_j) = \phi\{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$

   (a) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} = \emptyset$. Then

   $$\phi(x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1)$$

   $$= \{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$$

   $$= \{x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$$

   $$= \{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_i \bullet_{\rho_2} a_j = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$$

   (b) If $a_i \bullet_{\rho_1} a_j \cap \{a_i, a_j\} \neq \emptyset$. Assume e.g. that $a_i$ belongs to $a_i \bullet_{\rho_1} a_j$, then

   $$\phi(x \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1)$$

   $$= \{\phi(x) \in H : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\}$$

   $$= \{x \in H, x \neq a_i : (a_i, x) \in \rho_1 \text{ and } (x, a_j) \in \rho_1\} \cup \{a_j\}$$

   $$= \{x \in H : (a_j, x) \in \rho_2 \text{ and } (x, a_i) \in \rho_2\} = a_j \bullet_{\rho_2} a_i = \phi(a_i) \bullet_{\rho_2} \phi(a_j)$$
Similar is the proof of the rest cases, i.e. \( a_j \) to be in \( a_i \bullet_{p_1} a_j \) or both \( a_i, a_j \) to be in \( a_i \bullet_{p_2} a_j \). Also, since the principle of duality is valid [4], the dual statement holds, i.e. \( \phi(a_j \bullet_{p_1} a_i) = \phi(a_j) \bullet_{p_2} \phi(a_i) \).

2. If \( a_k, a_\lambda \notin \{a_i, a_j\} \) then

\[
\phi(a_k) \bullet_{p_1} \phi(a_\lambda) = \phi\{x \in H : (a_k, x) \in p_1 \text{ and } (x, a_\lambda) \in p_1\} =
\]

(a) if neither \( a_i \) nor \( a_j \) belongs to \( a_k \bullet_{p_1} a_\lambda \) then

\[
\phi\{x \in H : (a_k, x) \in p_1 \text{ and } (x, a_\lambda) \in p_1\} = \{\phi(x) \in H : (a_k, x) \in p_1 \text{ and } (x, a_\lambda) \in p_1\}
\]

\[
= \{x \in H : (a_k, x) \in p_1 \text{ and } (x, a_\lambda) \in p_1\}
\]

and since \( \phi(a_i) = a_j, \phi(a_j) = a_i \) this is equal to

\[
\{x \in H : (a_k, x) \in p_1 \text{ and } (x, a_\lambda) \in p_1\}
\]

or \( \{x \in H : (a_k, x) \in p_2 \text{ and } (x, a_\lambda) \in p_2\} \) which is

\[
a_k \bullet_{p_2} a_\lambda \text{ or } \phi(a_k) \bullet_{p_2} \phi(a_\lambda)
\]

Similar is the proof for the cases \( \phi(a_k) \bullet_{p_1} \phi(a_i), \phi(a_k) \bullet_{p_2} \phi(a_j) \) and their duals.

From the above proposition, the following theorem derives.

**Theorem 3.3.** If the Boolean matrix \( M_\sigma \) derives from \( M_\rho \) by interchanging rows and the corresponding columns, then the hypergroupoids \( H_\sigma \) and \( H_\rho \) are isomorphic.

The isomorphic classes of these hypergroupoids are not computed in [11]. These can be counted with a proper modification of the function `Good[]`, which will then return all the binary matrices that form a hypergroupoid. Thus, the above function changes in one of its lines and can be found in the appendix as a module of the package.

Check, for example, the three binary relations with matrices of size 2

\[
\text{In[4]:=} \ h2 = \text{Good1[2]}
\]

\[
\text{Out[4]=} \ \{\{\{0,1\},\{1,1\}\},\{\{1,1\},\{1,0\}\},\{\{1,1\},\{1,1\}\}\}
\]

We are able now to give a function that forms all \( n! \) isomorphisms of a given binary relation.

\[
\text{IsomorphTest1[a_List] := Module[\{p, a1\],}
\]

\[
\text{p = Permutations[Range[1, Length[a]]];}
\]

\[
\text{Return[Table[a1 = a;}
\]

\[
\text{a1 = ReplaceAll[a1, a1[[All, Table[j2,}
\]
Let us see the six permutations of the matrix

\[
M_\rho = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\]

which are defined by corresponding binary relations, that give isomorphic hypergroupoids:

\[
\text{In[5]:= IsomorphTest1[{{{1, 0, 1}, {1, 1, 0}, {0, 1, 1}}, {{1, 1, 0}, {0, 1, 1}, {1, 0, 1}}, }
\]

\[
\{\{1,1,0\}, \{0,1,1\}, \{1,0,1\}, \{1,0,1\}, \{1,1,0\}, \{0,1,1\},
\{1,0,1\}, \{1,1,0\}, \{0,1,1\}, \{1,1,0\}, \{0,1,1\}, \{1,0,1\}]]
\]

In order to count the number of the different nonisomorphic classes of hypergroupoids of order \(n\), a \(n\)–digit array, called cardinalities, is used by the program. Each time the routine encounters an isomorphic class, it drops it from variable \(h2\).

\text{Cardin[d_]:= Module[{h2, cardinalities, len, temp1, temp},}

\text{h2 = Good1[d];}

\text{cardinalities = Table[0, \{j1, 1, Factorial[d]\}];}

\text{While[Length[h2] > 0, temp = Union[IsomorphTest1[h2[[1]]]]; len = Length[Union[temp]]; cardinalities[[len]] = cardinalities[[len]] + 1; h2 = Complement[h2, temp]]; Return[cardinalities]}

Then we get

\text{In[6]:= Cardin[2]}
\text{Out[6]:= \{1, 1\}}

\text{In[7]:= Total[\%]}
\text{Out[7]:= 2}

\text{In[8]:= Cardin[3]}
\text{Out[8]:= \{2, 1, 5, 0, 0, 9\}}
In[9]:= Total[%]
Out[9]:= 17
In[10]:= Cardin[4]
Out[10]:= {2, 0, 1, 5, 0, 7, 0, 4, 0, 0, 0, 78,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 207}
In[11]:= Total[%]
Out[11]:= 304
In[12]:= Cardin[5]
Out[12]= {2, 0, 0, 0, 5, 0, 0, 0, 0, 13, 0, 1, 0, 0, 8,
0, 0, 0, 0, 78, 0, 0, 0, 3, 0, 0, 0, 0, 0, 152,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 42, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2206, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 18150}
In[13]:= Total[%]
Out[13]= 20660

So, there are 2, 17, 304 and 20660 isomorphic classes (I.C.) of orders 2, 3, 4 and 5 respectively. For example, for order 4 there are 2 I.C. of cardinality 1, 1 I.C. of cardinality 3, 5 I.C. of cardinality 4, 7 I.C. of cardinality 6, 4 I.C. of cardinality 8, 78 I.C. of cardinality 12 and 207 I.C. of cardinality 24. These 2 + 1 + 3 + 5 + 4 + 7 + 6 + 4 + 8 + 12 + 207 = 304 I.C. form the

\[2 \cdot 1 + 1 \cdot 3 + 5 \cdot 4 + 7 \cdot 6 + 4 \cdot 8 + 12 \cdot 207 = 6003\]

non–isomorphic hypergroupoids of order 4.

We also mention that there are \(2^n \times n\) binary matrices of size \(n\). We may count the non–isomorphic ones by simply changing the line

\[h2 = \text{Good}[n];\]

in the routine \(\text{Cardin[]}\) by the line

\[h2=\text{Tuples}[\text{Tuples}[0,1,n],n].\]

Then we get

In[14]:= Cardin[1]
Out[14]= {2}
In[15]:= Cardin[2]
Out[15]= {4, 6}
In[16]:= Total[%]
Out[16]= 10
In[17]:= Cardin[3]
Out[17]= {4, 2, 28, 0, 0, 70}
In[18]:= Total[%]
Out[18]= 104
In[19]:= Cardin[4]
Out[19]= {4, 0, 4, 28, 0, 32, 0, 16, 0, 0, 0, 496,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2464}
In[20]:= Total[%]
Out[20]= 3044

The integer sequence 2, 10, 104, 3044 etc. coincides with the integer sequence A000595, appeared in [10] and represents the number of non-isomorphic unlabeled binary relations on \( n \) nodes.

### 3.2. Weak Associativity

As proved in Section 2 above, the total hypergroup is the only hypergroupoid that fulfills the property of associativity. Thus, we checked a weaker property, which is called Weak Associativity:

\[
(a(bc) \cap(ab)c \neq \emptyset \text{ for all } a, b, c \in H.
\]

Having, up to this point, constructed all the hypergroupoids of order 2, 3, 4 and 5, we check the validity of this property to all of them and we count the ones that verify it. The package is given and explained in the appendix.

Its results are:

In[21]:= BinaryTest[2]
Out[21]= 3
In[22]:= BinaryTest[3]
Out[22]= 43
In[23]:= BinaryTest[4]
Out[23]= 2619
In[24]:= BinaryTest[5]
Out[24]= 602431

The counting of the hypergoupoids of orders \( n \geq 6 \) is time consuming, so we discontinued at \( n = 5 \).

### 4. Conclusions

This paper shows that the total hypergroup is the only hypergroup which can be produced by hypercomposition (1.1). Since it is a hypergoup it is also a semi-hypergroup and a quasihypergroup. No other semi- or quasi- hypergroups can be produced by (1.1). On the other hand there exist lots of hypergoupoids that can be produced by (1), the number of which is calculated with the use of Mathematica packages that are constructed for this purpose and consist part of the contents of this paper. The results of these calculations are given in the cumulative Table 1 below for the orders 2, 3, 4 and 5:
Table 1: Cumulative results

<table>
<thead>
<tr>
<th>Order →</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean matrices (BM)</td>
<td>16</td>
<td>512</td>
<td>65536</td>
<td>33554432</td>
</tr>
<tr>
<td>BM forming Hypergroupoids</td>
<td>3</td>
<td>73</td>
<td>6003</td>
<td>2318521</td>
</tr>
<tr>
<td>BM forming Weak-Associative Hypergroupoids</td>
<td>3</td>
<td>43</td>
<td>2619</td>
<td>602431</td>
</tr>
<tr>
<td>Nonisomorphic BM</td>
<td>10</td>
<td>104</td>
<td>3044</td>
<td>291968</td>
</tr>
<tr>
<td>Nonisomorphic BM forming Hypergroupoids</td>
<td>2</td>
<td>17</td>
<td>304</td>
<td>20660</td>
</tr>
</tbody>
</table>

5. The Mathematica package

The Mathematica package referred to in Section 3, is given below.

```mathematica
BeginPackage["BinaryTest"];
Clear["BinaryTest"];

BinaryTest::usage = "BinaryTest[n] counts the binary relations of dimension n that form a hypergroupoid. It also counts the Weak-associative binary hypergroupoids"

Begin["Private"];
Clear["BinaryTestPrivate"];

BinaryTest[n_] :=
Module[{c, ch},
c = Good1[n];
ch = Table[AssociativityWeakTest[HyperGroupoid[c[[j1]]]],
{j1, 1, Length[c]}];
Return[Count[ch, True]]];

Good1[n_] :=
Module[{c, i1, z},
c = Tuples[Tuples[{0, 1}, n], n];
z = Table[Min[Flatten[c[[i1]].c[[i1]]]],
{i1, 1, 2^(n*n)}];
Return[Select[Transpose[{c, z}], #2 > 0 &][All, 1]]
]

AssociativityWeakTest[a_List] :=
Module[{i, j, k, test},
i = 1; j = 1; k = 1; test = True;
While[test && i <= Length[a],
  test = Intersection[
```
Union[Flatten[Union[Extract[a, 
Distribute[{a[[i, j]], {k}, List]}]], 
Union[Flatten[Union[Extract[a, 
Distribute[{{i}, a[[j, k]]}, List]]]]] 
] != {};
k = k + 1;
If[k > Length[a], k = 1; j = j + 1;
If[j > Length[a], i = i + 1; j = 1];
 ];
Return[test]
];

HyperGroupoid[a_List] :=
Table[Table[Intersection[
Floor[(a[[j1, 1 ;; Length[a]]] 
+ a[[1 ;; Length[a], j2]])/2
]*
Table[j3, {j3, 1, Length[a]}],
Table[j3, {j3, 1, Length[a]}]
],
{j1, 1, Length[a]},
{j2, 1, Length[a]}];
End[];
EndPackage[];

The package consists of four functions. The three internal ones are:

**Good1**: that returns all the hypergroupoids associated to a binary relation of order \( n \).

**HyperGroupoid**: that constructs the hypergroupoid associated to a given Boolean matrix of a binary relation.

**AssociativityWeakTest**: that tests property (3.1) by forming all \( n^3 \) possible products of all triplets of elements of \( H \).

and the main one, which is:

**BinaryTest**: After calling Good, it constructs the table of the deriving hypergroupoids, using the function HyperGroupoid. Finally, it counts the number of those, which satisfy the property of the weak associativity.
References


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