LOWER AND UPPER BOUNDS OF THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral, in the monotonicity case of one function, are given. Applications in relation with the Steffensen generalisation of the Čebyšev inequality are provided.

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1. Introduction

In [3], S.S. Dragomir introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$
\begin{aligned}
(1.1)\quad T(f,g;u) :=& \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) \\
&- \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),
\end{aligned}
$$

provided $u(b)\neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

In order to bound the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of the integrals, as described in the definition of the Čebyšev functional (1.1), the first author obtained the inequality:

$$
\begin{aligned}
(1.2)\quad & |T(f,g;u)| \\
&\le \frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b (u),
\end{aligned}
$$

provided u is of bounded variation, f,g are continuous on $[a,b]$ and $m \leq f(t) \leq M$ for any $t \in [a,b]$. The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Moreover, if f,g are as above and u is monotonic nondecreasing on $[a,b]$, then

$$|T(f,g;u)| \leq \frac{1}{2}(M-m)\frac{1}{u(b)-u(a)}\int_a^b \left| g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| du(t), \tag{1.3}$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f and g are Riemann integrable and u is Lipschitzian with the constant $L > 0$, then also

$$|T(f,g;u)| \leq \frac{1}{2}L(M-m)\frac{1}{|u(b)-u(a)|}\int_a^b \left| g(t) - \frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right| dt, \tag{1.4}$$

provided $m \leq f(t) \leq M$, $t \in [a,b]$. The multiplicative constant $\frac{1}{2}$ is best possible in (1.4).

For results concerning bounds for the Čebyšev functional $T(f,g;u)$ see [4] and [5]. For other recent results on inequalities for the Riemann-Stieltjes integral, see [1], [2] and [6].

The main aim of this paper is to provide an upper and a lower bound for the functional $T(f,g;u)$ under the monotonicity assumption on the function f. An application for the Čebyšev inequality for Riemann-Stieltjes integrals that is related to Steffensen's result from [8] is given as well.

2. The results

The following result providing upper and lower bounds for the quantity

$$[h(b)-h(a)]\,T(f,g,h;a,b)$$

can be stated:

Theorem 2.1 *Let $f,g,h:[a,b]\to\mathbb{R}$ be such that $h(a)\neq h(b)$ and the Riemann-Stieltjes integrals $\int_a^b f(t)\,dh(t)$, $\int_a^b g(t)\,dh(t)$ and $\int_a^b f(t)\,g(t)\,dh(t)$ exist. If f*

is monotonic nondecreasing, then

$$(2.1)\quad [f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}$$
$$\leq \int_a^b f(t)\,g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)$$
$$\leq [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,dh(s)-\frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.$$

If f is monotonic nonincreasing, then:

$$(2.2)\quad [f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}$$
$$\leq \int_a^b f(t)\,g(t)\,dh(t)-\frac{1}{h(b)-h(a)}\cdot\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)$$
$$\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,dh(s)-\frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)\right\}.$$

Inequalities (2.1) *and* (2.2) *are sharp.*

Proof. We use the following Abel type inequality obtained by Mitrinović et al. in [7, p. 336]:

Let u be a nonnegative and monotonic nondecreasing function on $[a,b]$ and $v,w:[a,b]\to\mathbb{R}$ such that the Riemann-Stieltjes integrals $\int_a^b v(t)\,dw(t)$ and $\int_a^b u(t)\,v(t)\,dw(t)$ exist. Then

$$(2.3)\quad u(b)\inf_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}\leq\int_a^b u(t)\,v(t)\,dw(t)$$
$$\leq u(b)\sup_{t\in[a,b]}\left\{\int_t^b v(t)\,dw(t)\right\}.$$

We also use the representation (see [3])

$$(2.4)\quad T(f,g,h;a,b)$$
$$=\frac{1}{h(b)-h(a)}\int_a^b[f(t)-\gamma]\left[g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right]dh(t),$$

which holds for any $\gamma\in\mathbb{R}$.

Now, if we choose $\gamma=f(a)$, then we observe that the function $u(t)=f(t)-f(a)$ is nonnegative and monotonic nondecreasing on $[a,b]$ and applying

(2.3) for $w(t) = h(t)$ and $v(t) = g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(s)\, dh(s)$ we deduce:

$$\begin{aligned}
(2.5)\qquad & [f(b) - f(a)] \inf_{t\in[a,b]} \left\{ \int_t^b \left[g(s) - \frac{1}{h(b) - h(a)} \cdot \int_a^b g(\tau)\, dh(\tau) \right] dh(s) \right\} \\
& \le [h(b) - h(a)]\, T(f, g, h; a, b) \\
& \le [f(b) - f(a)] \sup_{t\in[a,b]} \left\{ \int_t^b \left[g(s) - \frac{1}{h(b) - h(a)} \cdot \int_a^b g(\tau)\, dh(\tau) \right] dh(s) \right\},
\end{aligned}$$

which is equivalent with the desired inequality (2.1).

For the second inequality, we use (2.4) with $\gamma = f(b)$ and the following Abel type result for functions u which are monotonic nonincreasing and nonnegative (see [7, p. 336]):

$$\begin{aligned}
(2.6)\qquad u(a) \inf_{t\in[a,b]} \left\{ \int_a^t v(t)\, dw(t) \right\} & \le \int_a^b u(t)\, v(t)\, dw(t) \\
& \le u(a) \sup_{t\in[a,b]} \left\{ \int_a^t v(t)\, dw(t) \right\}.
\end{aligned}$$

The details are omitted.

Let us prove for instance the sharpness of the second inequality in (2.1).

If we choose $h(t) = t$ and $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$ then we have to show that the inequality:

$$(2.7)\qquad \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \le [f(b) - f(a)] \sup_{t\in[a,b]} \left\{ \int_t^b \operatorname{sgn}\left(s - \frac{a+b}{2}\right) ds \right\}$$

is sharp provided f is monotonic nondecreasing on $[a, b]$.

Notice that

$$\lambda(t) := \int_t^b \operatorname{sgn}\left(s - \frac{a+b}{2}\right) ds = \begin{cases} t - a & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\[2mm] b - t & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

and then $\sup_{t\in[a,b]} \lambda(t) = \frac{b-a}{2}$.

Therefore (2.7) becomes

$$(2.8)\qquad \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \le [f(b) - f(a)] \cdot \frac{b-a}{2}.$$

Now, if in this inequality we choose $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, which is monotonic nondecreasing on $[a, b]$, we get in both sides of (2.8) the same quantity $b - a$.

The sharpness of the other inequalities can be shown in a similar way. The details are omitted. ∎

Remark 2.1 We observe that

$$\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)$$
$$= \int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\cdot\left[\int_a^t g(s)\,dh(s)+\int_t^b g(s)\,dh(s)\right]$$
$$= \frac{h(t)-h(a)}{h(b)-h(a)}\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\int_a^t g(s)\,dh(s)$$
$$= \frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}$$
$$\times\left[\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s) - \frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s)\right].$$

Therefore, if we denote by $\Delta(g,h;t,a,b)$ the difference

$$\frac{1}{h(b)-h(t)}\int_t^b g(s)\,dh(s) - \frac{1}{h(t)-h(a)}\int_a^t g(s)\,dh(s),$$

provided $h(t)\neq h(a), h(b)$ for $t\in[a,b]$, then from (2.1) we get

$$[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\} \tag{2.9}$$
$$\le \int_a^b f(t)g(t)\,dh(t) - \frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t)$$
$$\le [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},$$

provided f is monotonic nondecreasing on $[a,b]$.

A similar result can be stated from (2.2) on noticing that

$$\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(\tau)\,dh(\tau)$$
$$= -\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b).$$

Indeed, since

$$\inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\cdot\int_a^b g(s)\,dh(s)\right\}$$
$$= \inf_{t\in[a,b]}\left(\sup_{t\in[a,b]}\right)\left\{-\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}$$
$$= -\sup_{t\in[a,b]}\left(\inf_{t\in[a,b]}\right)\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\},$$

then from (2.2) we get

$$\begin{aligned}
&[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}\\
&\leq \frac{1}{h(b)-h(a)}\int_a^b f(t)\,dh(t)\int_a^b g(t)\,dh(t)-\int_a^b f(t)g(t)\,dh(t)\\
&\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{[h(t)-h(a)][h(b)-h(t)]}{h(b)-h(a)}\Delta(g,h;t,a,b)\right\}
\end{aligned}$$

provided that f is monotonic nonincreasing on $[a,b]$.

The following corollary gives a particular result of interest for Riemann weighted integrals.

Corollary 2.1 *Let $f,g,w:[a,b]\to\mathbb{R}$ be such that the Riemann integrals $\int_a^b f(t)w(t)\,dt$, $\int_a^b g(t)w(t)\,dt$, $\int_a^b f(t)g(t)w(t)\,dt$ and $\int_a^b w(t)\,dt$ exist, and $\int_a^b w(t)\,dt\neq 0$.*

If f is monotonic nondecreasing, then

$$\begin{aligned}
(2.10)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)w(\tau)\,d\tau\right\}\\
&\leq \int_a^b f(t)g(t)w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)w(t)\,dt\cdot\int_a^b g(t)w(t)\,dt\\
&\leq [f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)w(s)\,ds-\frac{\int_t^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)w(\tau)\,d\tau\right\}.
\end{aligned}$$

If f is monotonic nonincreasing, then

$$\begin{aligned}
(2.11)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)w(\tau)\,d\tau\right\}\\
&\leq \int_a^b f(t)g(t)w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)w(t)\,dt\cdot\int_a^b g(t)w(t)\,dt\\
&\leq [f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)w(s)\,ds-\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\int_a^b g(\tau)w(\tau)\,d\tau\right\}.
\end{aligned}$$

Remark 2.2 If we define

$$\tilde{\Delta}(g,w;t,a,b):=\frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)w(s)\,ds-\frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)w(s)\,ds,$$

provided $\int_a^t w(s)\,ds,\ \int_t^b w(s)\,ds \neq 0$, then, under the assumptions of Corollary 2.1, we have:

$$\begin{aligned}(2.12)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\\ &\le\int_a^b f(t)\,g(t)\,w(t)\,dt-\frac{1}{\int_a^b w(s)\,ds}\cdot\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt\\ &\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\},\end{aligned}$$

provided f is monotonic nondecreasing on $[a,b]$, and

$$\begin{aligned}(2.13)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\\ &\le\frac{1}{\int_a^b w(s)\,ds}\int_a^b f(t)\,w(t)\,dt\cdot\int_a^b g(t)\,w(t)\,dt-\int_a^b f(t)\,g(t)\,w(t)\,dt\\ &\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\frac{\int_a^t w(s)\,ds\int_a^b w(s)\,ds}{\int_a^b w(s)\,ds}\cdot\tilde{\Delta}(g,w;t,a,b)\right\}\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

Remark 2.3 In the particular case where $w(t)=1$, $t\in[a,b]$, we get the simpler inequalities:

$$\begin{aligned}(2.14)\quad &[f(b)-f(a)]\inf_{t\in[a,b]}\left\{\int_t^b g(s)\,ds-\frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\le\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\le[f(b)-f(a)]\sup_{t\in[a,b]}\left\{\int_t^b g(s)\,ds-\frac{b-t}{b-a}\int_a^b g(\tau)\,d\tau\right\}\end{aligned}$$

in the case where f is monotonic nondecreasing on $[a,b]$.

If f is monotonic nonincreasing on $[a,b]$, then

$$\begin{aligned}(2.15)\quad &[f(a)-f(b)]\inf_{t\in[a,b]}\left\{\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}\\ &\le\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\ &\le[f(a)-f(b)]\sup_{t\in[a,b]}\left\{\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(\tau)\,d\tau\right\}.\end{aligned}$$

If we denote

$$\bar{\Delta}(g;t,a,b) := \frac{1}{b-t}\int_t^b g(s)\,ds - \frac{1}{t-a}\int_a^t g(s)\,ds,$$

then we get from (2.14)

$$\begin{aligned}
&\frac{[f(b)-f(a)]}{b-a}\inf_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}\\
&\le \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt\\
&\le \frac{[f(b)-f(a)]}{b-a}\sup_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\},
\end{aligned}$$

provided f is monotonic nondecreasing and from (2.15)

$$\begin{aligned}
&\frac{[f(a)-f(b)]}{b-a}\inf_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}\\
&\le \frac{1}{b-a}\int_a^b f(t)\,dt\cdot\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\\
&\le \frac{[f(a)-f(b)]}{b-a}\sup_{t\in[a,b]}\left\{(t-a)(b-t)\bar{\Delta}(g;t,a,b)\right\}
\end{aligned}$$

if f is monotonic nonincreasing on $[a,b]$.

3. Applications for the Čebyšev inequality

Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p:[a,b]\to[0,\infty)$ be an integrable function, then [7, p. 239]:

$$\int_a^b p(x)\,dx\int_a^b p(x)\,f(x)\,g(x)\,dx \ge \int_a^b p(x)\,f(x)\,dx\int_a^b p(x)\,g(x)\,dx. \tag{3.1}$$

This inequality in known in the literature as Čebyšev's inequality.

For various other results related to this classical fact, see Chapter IX of the book [7] .

Proposition 3.1 *Let $f,g,h:[a,b]\to\mathbb{R}$ be such that the Riemann-Stieltjes integrals $\int_a^b f(t)\,dh(t)$, $\int_a^b g(t)\,dh(t)$ and $\int_a^b f(t)\,g(t)\,dh(t)$ exist. If $h(b)>h(a)$, f is monotonic nondecreasing (nonincreasing) and*

$$[h(b)-h(a)]\int_a^t g(s)\,dh(s) \ge [h(b)-h(t)]\int_a^b g(s)\,dh(s) \tag{3.2}$$

for any $t\in[a,b]$, then

$$[h(b)-h(a)]\int_a^b f(t)\,g(t)\,dh(t) \ge (\le) \int_a^b f(t)\,dh(t)\cdot\int_a^b g(t)\,dh(t). \tag{3.3}$$

The proof follows by Theorem 2.1 by using

$$\int_t^b g(s)\,dh(s) - \frac{h(b)-h(t)}{h(b)-h(a)}\int_a^b g(s)\,dh(s) = -\left[\int_a^t g(s)\,dh(s) - \frac{h(t)-h(a)}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right].$$

Remark 3.4 The above proposition implies the following Čebyšev type inequality for weighted integrals (with not necessarily positive weights).

Let $f, g, w : [a,b] \to \mathbb{R}$ be such that the Riemann integrals, $\int_a^b w(t)\,dt$, $\int_a^b f(t)\,w(t)\,dt$, $\int_a^b g(t)\,w(t)\,dt$ and $\int_a^b f(t)\,g(t)\,w(t)\,dt$ exist.

If $\int_a^b w(t)\,dt > 0$, f is monotonic nondecreasing (nonincreasing) and

$$\int_a^b w(s)\,ds \int_t^b g(s)\,w(s)\,ds \geq \int_t^b w(s)\,ds \int_a^b g(s)\,w(s)\,ds \tag{3.4}$$

for any $t \in [a,b]$, then

$$\int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt \geq (\leq) \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt. \tag{3.5}$$

In particular (i.e., if $w(s) = 1$), if f is monotonic nondecreasing (nonincreasing) and if

$$(b-a)\int_a^t g(s)\,ds \geq (b-t)\int_a^b g(s)\,ds \tag{3.6}$$

for any $t \in [a,b]$, then

$$(b-a)\int_a^b f(t)\,g(t)\,dt \geq (\leq) \int_a^b f(t)\,dt \int_a^b g(t)\,dt. \tag{3.7}$$

Remark 3.5 Notice that, the weighted inequality (3.5), as pointed out in [7, p. 246], can be also obtained from the Steffensen result [8] which states that: if F, G, H are integrable functions on $[a,b]$ such that for all $x \in [a,b]$

$$\frac{\int_a^x G(t)\,dt}{\int_a^b G(t)\,dt} \leq \frac{\int_a^x H(t)\,dt}{\int_a^b H(t)\,dt},$$

then

$$\frac{\int_a^b F(t)\,G(t)\,dt}{\int_a^b G(t)\,dt} \geq \frac{\int_a^b F(t)\,H(t)\,dt}{\int_a^b H(t)\,dt}, \tag{3.8}$$

provided F is monotonic nondecreasing on $[a, b]$.

The choice $F(t) \equiv f(t)$, $H(t) = w(t)$, and $G(t) = g(t)w(t)$ in (3.8) produces (3.5) under the condition that (3.4) holds and f is monotonic nondecreasing.

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