NOTES FOR HARTLEY TRANSFORMS OF GENERALIZED FUNCTIONS

S.K.Q. Al-Omari

Department of Applied Sciences
Faculty of Engineering Technology
Al-Balqa Applied University
Amman 11134
Jordan
e-mail: s.k.q.alomari@fet.edu.jo

Abstract. The classical Hartley transform, originally introduced by Hartley as a real transform with a number of properties being similar to the properties of Fourier transform. In this work, we extend the Hartley transform to certain space of distributions of compact support. Further, we establish that the Hartley transform and its inverse are one to one and onto mappings in the space of Boehmians. Moreover, continuity with respect to δ and Δ convergence is discussed in some detail. Certain theorems are also proved.

Keywords: Hartly transform; Boehmian space; convolution; smooth function; distribution of compact support.

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1. Introduction

The Hartley transform is a spectral transform closely related to the Fourier transforms. It contain the same information that the Fourier transform does, and no advantage accrues in its use for complex signals. However, for real signal, the Hartley transform is real and this can offer computational advantages in signal processing applications that traditionally make use of Fourier transforms [15]. Moreover, the Hartley transform can be analytically continued into the complex plane, and for real functions it is Hermitian symmetry on reflection in the real axis. The Hartley transform in the complex plane is an entire function of exponential type with zeros close to the real axis, a property shared with Fourier transform [15, p.p. 414]. The Hartley transform $H(\upsilon)$, of $f(x)$, is defined by [13]-[17]

$$H(\upsilon) = \int_{\mathbb{R}} f(x)\, cas\,(2\pi x\upsilon)\, dx, \tag{1}$$

where

$$cas\,(2\pi\upsilon x) = \cos(2\pi\upsilon x) + \sin(2\pi\upsilon x).$$

Therefore, it follows that the inverse Hartley transform is given by

$$f(x) = \int_{\mathbb{R}} H(v)\, cas(2\pi x v)\, dv. \tag{2}$$

The scalling and linearity conditions of the Hartley transform have been described in [16]. Theorems for the Hartley transform, analogous to those for the Fourier transform, can be easily derived from definitions. The more complicated, compared to the Fourier transform, is the convolution theorem which can sometimes amount to a disadvantage of the Hartley transform. For functions f and g, L^1 functions, the convolution theorem of the Hartley is defined by

$$H(f \bullet g)(v) = \frac{1}{2}\boldsymbol{G}(Hf \times Hg)(v), \tag{3}$$

where

$$\boldsymbol{G}(f \times g)(v) = f(v)g(v) + f(v)g(-v) + f(-v)g(v) - f(-v)g(-v). \tag{4}$$

and $\bullet$ is the usual convolution product.

2. Boehmian spaces

The general construction of Boehmians is algebraic in nature. Boehmians were first constructed as a generalization of regular Mikusinski operators [5]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(i) A nonempty set $\mathbb{X}$ and a commutative semigroup $(\mathbb{Y}, *)$;

(ii) An operation $\circ : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ such that for each $x \in \mathbb{X}$ and $s_1, s_2, \in \mathbb{Y}$, $x \circ (s_1 * s_2) = (x \circ s_1) \circ s_2$;

(iii) A collection $\Delta \subset \mathbb{Y}^N$ such that:

  a) If $x, y \in \mathbb{X}, (s_n) \in \Delta$, $x \circ s_n = y \circ s_n$ for all n, then $x = y$;
  b) If $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Elements of Δ are called delta sequences.

Let $\digamma = \{(x_n, s_n) : x_n \in \mathbb{X}, s_n \in \Delta, x_n \circ s_m = x_m \circ s_n, \forall m, n \in \mathbf{N}\}$. Consider $(x_n, s_n), (y_n, t_n) \in \digamma, x_n \circ t_m = y_m \circ s_n, \forall m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in $\digamma$. The space of equivalence clases in $\digamma$ is denoted by β. Elements of β are called Boehmians. A typical element of β is written as $\left[\frac{x_n}{s_n}\right]$. Between $\mathbb{X}$ and β there is a canonical embedding expressed as $x \to \frac{x \circ s_n}{s_n}$. The operation $\circ$ can be extended to $\beta \times \mathbb{X}$ by $\frac{x_n}{s_n} \circ t = \frac{x_n \circ t}{s_n}$. In β, two types of convergence:

Type 1: *A sequence* (h_n) *in* β *is said to be* δ **convergent** *to* h *in* β, *denoted by* $h_n \xrightarrow{\delta} h$, *if there exists a delta sequence* (s_n) *such that* $(h_n \circ s_n), (h \circ s_n) \in \mathbb{X}$, $\forall k, n \in \mathbf{N}$, *and* $(h_n \circ s_k) \to (h \circ s_k)$ *as* $n \to \infty$,*in* $\mathbb{X}$, *for every* $k \in \mathbf{N}$.

Type 2: *A sequence* (h_n) *in* β *is said to be* Δ **convergent** *to* h *in* β, *denoted by* $h_n \xrightarrow{\Delta} h$, *if there exists a* $(s_n) \in \Delta$ *such that* $(h_n - h) \circ s_n \in \mathbb{X}, \forall n \in \mathbf{N}$, *and* $(h_n - h) \circ s_n \to 0$ *as* $n \to \infty$ *in* $\mathbb{X}$. For more details, see [1]-[5], [7]-[9], [11], [12].

3. The Boehmian Space $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$

By $\mathbb{D}(\mathbb{R})$, we denote the space of test functions of compact support and $\mathbb{E}(\mathbb{R})$ the space of all infinitely smooth functions on $\mathbb{R}$ equipped with the sequence of multinorms $\xi_k(f) = \sup_{x \in K} \left| f^{(k)}(x) \right|$ for every $f \in \mathbb{E}(\mathbb{R})$, where K run through compact subsets of $\mathbb{R}$. The kernel function of the Hartley transform, $cas 2\pi x \upsilon$, is certainly a member of $\mathbb{E}(\mathbb{R})$ for arbitrary but fixed υ. This justifies the extension of the Hartley transform to the context of distributions through the formula

$$Hf(\upsilon) = \langle f(x), cas 2\pi \upsilon x \rangle$$

for each $f \in \mathbb{E}'(\mathbb{R})$, the strong dual of $\mathbb{E}(\mathbb{R})$ of distriutions of compact support [10], [19]. Let Δ be the family of sequences (γ_n) from $\mathbb{D}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \gamma_n(x)\, dx = 1, \qquad \text{for every } n \in \mathbf{N}. \tag{5}$$

$$\int_{\mathbb{R}} |\gamma_n(x)|\, dx \le M, \qquad \text{for some positive } M. \tag{6}$$

$$\operatorname{supp} \gamma_n(x) \subset (-\varepsilon_n, \varepsilon_n), \qquad \varepsilon_n \to 0 \text{ as } n \to \infty. \tag{7}$$

Members of Δ are named as delta sequences .

Lemma 3.1. *Let* $f \in \mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f \bullet \phi \in \mathbb{E}(\mathbb{R})$.

Lemma 3.2. *Given* $f_1, f_2 \in \mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then for every* $\alpha \in \mathbb{C}$,

$$(f_1 + f_2) \bullet \phi = f_1 \bullet \phi + f_2 \bullet \phi \text{ and } \alpha(f_1 \bullet \phi) = (\alpha f_1) \bullet \phi = f_1 \bullet (\alpha \phi).$$

Lemma 3.3. *Let* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f_n \bullet \phi \to f \bullet \phi$.

Proofs of Lemmas 3.1-3.3 are straightforward from the properties of the integral operator.

Lemma 3.4. *Let* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \to \infty$ *and* $\phi \in \mathbb{D}(\mathbb{R})$ *then* $f_n \bullet \phi \to f$ *as* $n \to \infty$.

Proof. If $f_n \to f$ in $\mathbb{E}(\mathbb{R})$ then from Eq. (5) we get $\left| D^k (f_n \bullet \phi - f)(x) \right| \le M \xi_k (f_n(x-t) - f(x)) \to 0$ as $n \to \infty$. Hence $\xi_k (f_n \bullet \phi - f) \to 0$ as $n \to \infty$. The lemma is completely proved.

The Boehmian Space $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ is constructed. The sum of two Boehmians and multiplication by a scalar in $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ can be defined in a natural way $\left[\frac{f_n}{\phi_n}\right]+\left[\frac{g_n}{\psi_n}\right]=\left[\frac{f_n\bullet\psi_n+g_n\bullet\phi_n}{\phi_n\bullet\psi_n}\right]$ and $\alpha\left[\frac{f_n}{\phi_n}\right]=\left[\alpha\frac{f_n}{\phi_n}\right], \alpha\in\mathbb{C}$. The operation $\bullet$ and the differentiation are defined by $\left[\frac{f_n}{\phi_n}\right]\bullet\left[\frac{g_n}{\psi_n}\right]=\left[\frac{f_n\bullet g_n}{\phi_n\bullet\psi_n}\right]$ and $D^\alpha\left[\frac{f_n}{\phi_n}\right]=\left[\frac{D^\alpha f_n}{\phi_n}\right]$. The relationship between the notion of convergence and the product $\bullet$ is given by:

1$-$If $f_n\to f$ as $n\to\infty$ in $\mathbb{E}(\mathbb{R})$ and, $\phi\in\mathbb{D}(\mathbb{R})$ is fixed, then $f_n\bullet\phi\to f\bullet\phi$ in $\mathbb{E}(\mathbb{R})$, as $n\to\infty$; 2$-$ If $f_n\to f$ as $n\to\infty$ in $\mathbb{E}(\mathbb{R})$ and $(\delta_n)\in\Delta$, then $f_n\bullet\delta_n\to f$ in $\mathbb{E}(\mathbb{R})$, as $n\to\infty$. The operation $\bullet$ can be extended to $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)\times\mathbb{D}$ in the sense that *If* $[f_n/\delta_n]\in\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ *and* $\phi\in\mathbb{D}(\mathbb{R})$,*then* $[\frac{f_n}{\phi_n}]\bullet\phi=[\frac{f_n\bullet\phi}{\delta_n}]$.

Convergence in $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$, is defined as follows: *A sequence of* (β_n) *in* $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ *is said to be* δ *convergent to a Boehmian* β *in* $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$, *denoted by* $\beta_n\xrightarrow{\delta}\beta$, *if there exists a delta sequence* (δ_n) *such that* $\beta_n\bullet\delta_n,\beta\bullet\delta_n\in\mathbb{E}(\mathbb{R}),\forall k,n\in\mathbf{N}$, *and* $\beta_n\bullet\delta_k\to\beta\bullet\delta_k$ *as* $n\to\infty$, *in* $\mathbb{E}(\mathbb{R})$, *for every* $k\in\mathbf{N}$. This can be interpreted to mean: $\beta_n\xrightarrow{\delta}\beta\ (n\to\infty)$ *in* $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ *if and only if there is* $f_{n,k},f_k\in\mathbb{E}(\mathbb{R})$ *and* $(\delta_k)\in\Delta$ *such that* $\beta_n=\left[\frac{f_{n,k}}{\delta_k}\right],\beta=\left[\frac{f_k}{\delta_k}\right]$ *and for each* $k\in\mathbf{N}, f_{n,k}\to f_k$ as $n\to\infty$ in $\mathbb{E}(\mathbb{R})$. It is more often convenient to use another kind of convergence**:**

A sequence (β_n) *in* $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$ *is said to be* Δ *convergent to a* β *in* $\mathbb{M}(\mathbb{E},\mathbb{D},\Delta,\bullet)$, *denoted by* $\beta_n\xrightarrow{\Delta}\beta$, *if there exists a* $(\delta_n)\in\Delta$ *such that* $(\beta_n-\beta)\bullet\delta_n\in\mathbb{E}(\mathbb{R}),\forall n\in\mathbf{N}$, *and* $(\beta_n-\beta)\bullet\delta_n\to 0$ *as* $n\to\infty$ *in* $\mathbb{E}(\mathbb{R})$.

4. The Space $\mathbb{M}_H\left(\mathbb{E},\mathbb{D}^H,\Delta^H,\curlyvee\right)$

To extend the Hartley transform to Boehmians we describe another space of Boehmians as follows. First, it will be necessary to know that, if $(\gamma_n)\in\Delta$ then

$$H\gamma_n(\upsilon)\to 1 \text{ as } n\to\infty. \tag{8}$$

and that

$$H\gamma_n(-\upsilon)\to 1 \text{ as } n\to\infty. \tag{9}$$

uniformly on compact subsets. Let a mapping $\curlyvee$ between f and $H\phi$ be defined by

$$(f\curlyvee H\phi)(\upsilon)=\frac{1}{2}\boldsymbol{G}(f\times H\phi)(\upsilon). \tag{10}$$

$\boldsymbol{G}$ has the usual meaning in (4). Denote by $\mathbb{D}^H(\mathbb{R})$, the set of all Hartley transform of functions from $\mathbb{D}(\mathbb{R})$. Define $\Delta^H=\{H\gamma_n:\gamma_n\in\Delta,\ \forall n\in\mathbf{N}\}$. We prove the following

Lemma 4.1. *Let* $f\in\mathbb{E}(\mathbb{R}),H\phi\in\mathbb{D}^H(\mathbb{R})$ *then* $(f\curlyvee H\phi)(\upsilon)\in\mathbb{E}(\mathbb{R})$.

Proof. Let $k \in \mathbf{N}$ then for each $f \in \mathbb{E}(\mathbb{R})$ and $H\phi \in \mathbb{D}^H \subset \mathbb{E}(\mathbb{R})$ we have $H\phi(\mp\upsilon) f(\pm\upsilon) \in \mathbb{E}(\mathbb{R})$. If K is a compact subset of $\mathbb{R}$ containing supp ϕ then by using Eq. (10) and Eq. (4) we get

$$\sup_{\upsilon \in K} \left| D_\upsilon^k (f \curlyvee H\phi)(\upsilon) \right| < \infty. \tag{11}$$

Allowing K traverses the set of real numbers yields $f \curlyvee H\phi \in \mathbb{E}(\mathbb{R})$. This completes the proof of the lemma.

Lemma 4.2. *A mapping* $\mathbb{E} \times \mathbb{D}^H \to \mathbb{E}$ *defined by*

$$(f, H\phi) \to f \curlyvee H\phi \tag{12}$$

satisfies the following

(i) *If* $H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$ then $(H\phi \curlyvee H\psi)(\upsilon) \in \mathbb{D}^H(\mathbb{R})$.

(ii) *If* $f, g \in \mathbb{E}(\mathbb{R}), H\phi \in \mathbb{D}^H(\mathbb{R})$ *then* $((f+g) \curlyvee H\phi)(\upsilon) = (f \curlyvee H\phi)(\upsilon) + (g \curlyvee H\psi)(\upsilon)$.

(iii) $(H\phi \curlyvee H\psi)(\upsilon) = (H\psi \curlyvee H\phi)(\upsilon), \forall H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$.

(iv) *If* $f \in \mathbb{E}(\mathbb{R}), H\phi, H\psi \in \mathbb{D}^H(\mathbb{R})$ *then* $(f \curlyvee H\phi) \curlyvee H\psi = f \curlyvee (H\phi \curlyvee H\psi)$.

Proof. (i) Let $\upsilon \in \mathbb{R}$ then it is clear that $(H\phi \curlyvee H\psi)(\upsilon) = H(\phi \bullet \psi)(\upsilon)$. But Lemma 3.1 implies $\phi \bullet \psi \in \mathbb{D}(\mathbb{R})$. Hence $H\phi \curlyvee H\psi \in \mathbb{D}^H(\mathbb{R})$.

(ii) is obvious.

(iii) Since $\phi \bullet \psi \in \mathbb{D}(\mathbb{R}), \phi \bullet \psi = \psi \bullet \phi$. Applying the Hartley transform yields $H(\psi \bullet \phi)(\upsilon) = H(\phi \bullet \psi)(\upsilon)$. This implies $H\phi \curlyvee H\psi = H\psi \curlyvee H\phi$.

(iv) can be easily established by routine calculation from Eq. (4) and Eq. (10). The lemma is completely proved.

Lemma 4.3. *Let* $f_1, f_2 \in \mathbb{E}(\mathbb{R}), H\gamma_n \in \Delta^H, \forall n, f_1 \curlyvee H\gamma_n = f_2 \curlyvee H\gamma_n, \forall n,$ *then* $f_1 = f_2$ *in* $\mathbb{E}(\mathbb{R})$.

Proof. From hypothesis $f_1 \curlyvee H\gamma_n = f_2 \curlyvee H\gamma_n, \forall n$. Invoking Eq. (8) and Eq. (9) in Eq. (10), yields $f_1(\upsilon) = f_2(\upsilon), \forall \upsilon \in \mathbb{R}$. Thus $f = g$. This completes the proof of the lemma.

Lemma 4.4.

(1) *If* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \to \infty$ *and* $H\phi \in \mathbb{D}^H(\mathbb{R})$ *then* $f_n \curlyvee H\phi \to f \curlyvee H\phi$ *as* $n \to \infty$.

(2) *If* $f_n \to f$ *in* $\mathbb{E}(\mathbb{R})$ *as* $n \to \infty$ *and* $H\gamma_n \in \Delta^H$ *then* $f_n \curlyvee H\gamma_n \to f$ *as* $n \to \infty$ *in* $E(\mathbf{R})$.

The proof of the above lemma is a result of Eq. (8) and Eq. (9).

Lemma 4.5. *Let* $(H\gamma_n)_1^\infty, (H\xi_n)_1^\infty \in \Delta^H$ *then* $(H\gamma_n \curlyvee H\xi_n)_1^\infty \in \Delta^H$.

Proof. Let $(\gamma_n), (\xi_n) \in \Delta$ then $(\gamma_n \bullet \xi_n) \in \Delta$. Thus, $(\gamma_n \bullet \xi_n) \in \Delta$ is the sequence such that $(H\gamma_n \curlyvee H\xi_n)(\upsilon) = H(\gamma_n \bullet \xi_n)(\upsilon) \in \Delta^H$. Hence the lemma is completely proved. Further, it can be observed that

$$H\gamma_n \curlyvee H\xi_n \to 1 \text{ as } n \to \infty,$$

for every $(\gamma_n), (\xi_n) \in \Delta$. Hence, Δ^H satisfies the necessary conditions for delta sequences. The Boehmian space $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$, or $\mathbb{M}_H$, is constructed. Addition, scalar multiplication, differentiation, convolution and convergence can be defined in a natural way. For some detail, the sum and multiplication by a scalar on $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ is defined as $\left[\frac{Hf_n}{H\phi_n}\right] + \left[\frac{Hg_n}{H\psi_n}\right] = \left[\frac{Hf_n \curlyvee H\psi_n + Hg_n \curlyvee H\phi_n}{H\phi_n \curlyvee H\psi_n}\right]$ and $\alpha\left[\frac{Hf_n}{H\phi_n}\right] = \left[\alpha\frac{Hf_n}{H\phi_n}\right], \alpha \in \mathbb{C}$. The operation $\curlyvee$ and the differentiation are defined by $\left[\frac{Hf_n}{H\phi_n}\right] \curlyvee \left[\frac{Hg_n}{H\psi_n}\right] = \left[\frac{Hf_n \curlyvee Hg_n}{H\phi_n \curlyvee H\psi_n}\right]$ and $D^\alpha\left[\frac{Hf_n}{H\phi_n}\right] = \left[\frac{D^\alpha Hf_n}{H\phi_n}\right]$. Convergence on $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ is defined by:

A sequence of $(H\beta_n)$ *in* $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *is said to be* δ **convergent** *to a Boehmian* $H\beta$, *denoted by* $H\beta_n \xrightarrow{\delta} H\beta$, *if there exists a delta sequence* $(H\delta_n)$ *such that* $H\beta_n \curlyvee H\delta_n, H\beta \curlyvee H\delta_n \in \mathbb{E}(\mathbb{R}), \forall k, n \in \mathbf{N}$, *and* $H\beta_n \curlyvee H\delta_k \to H\beta \curlyvee H\delta_k$ *as* $n \to \infty$, *in* $\mathbb{E}(\mathbb{R})$, *for every* $k \in \mathbf{N}$. This can be interpreted to mean: $H\beta_n \xrightarrow{\delta} H\beta$ $(n \to \infty)$ *if and only if there is* $Hf_{n,k}, Hf_k \in \mathbb{E}(\mathbb{R})$ *and* $(H\delta_k) \in \Delta^H$ *such that* $H\beta_n = \left[\frac{Hf_{n,k}}{H\delta_k}\right], H\beta = \left[\frac{Hf_k}{H\delta_k}\right]$ *and for each* $k \in \mathbf{N}, Hf_{n,k} \to Hf_k$ as $n \to \infty$ in $\mathbb{E}(\mathbb{R})$.

It is more often convenient to use another kind of convergence: *A sequence* $(H\beta_n) \in \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *is said to be* Δ **convergent** *to a* $H\beta \in \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$, *denoted by* $H\beta_n \xrightarrow{\Delta} H\beta$, *if there exists a* $(H\delta_n) \in \Delta^H$ *such that* $(H\beta_n - H\beta) \curlyvee H\delta_n \in \mathbb{E}(\mathbb{R})$, $\forall n \in \mathbf{N}$, *and* $(H\beta_n - H\beta) \curlyvee H\delta_n \to 0$ *as* $n \to \infty$ *in* $\mathbb{E}(\mathbb{R})$.

Theorem 4.6. *The mapping*

$$\begin{array}{lcl} \mathbb{E} & \to & \mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee) \\ f & \to & \left[\frac{f \curlyvee H\gamma_n}{H\gamma_n}\right] \end{array} \tag{13}$$

is a continuous imbedding of $\mathbb{E}(\mathbb{R})$ *into* $\mathbb{M}_H(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee)$ *with respect to* δ *convergence.*

Proof. To show the mapping is one to one let $\left[\frac{f \curlyvee H\gamma_n}{H\gamma_n}\right] = \left[\frac{g \curlyvee Ht_n}{Ht_n}\right]$. Then

$$(f \curlyvee H\gamma_n) \curlyvee Ht_m = (g \curlyvee Ht_m) \curlyvee H\gamma_n.$$

For large values of m and n, $Ht_m, H\gamma_n \to 1$. The above equation is therefore reduced to $f = g$. To establish continuity of Eq. (13) with respect to $\delta-$convergence, let $f_n \to 0$ as $n \to \infty$ then $f_n \curlyvee H\gamma_n \to 0$ as $n \to \infty$. Hence $\left[\frac{f_n \curlyvee H\gamma_n}{H\gamma_n}\right] \to 0$ as $n \to \infty$. The theorem is completely proved.

5. Hartley transform of Boehmians

Let $\left[\frac{f_n}{\gamma_n}\right] \in \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ then, in view of analysis established in Section 4, we define the extended Hartley transform by

$$\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{Hf_n}{H\gamma_n}\right]. \tag{14}$$

in $\mathbb{M}\left(\mathbb{E}, \mathbb{D}^H, \Delta^H, \curlyvee\right)$.

Theorem 5.1. $\boldsymbol{\kappa} : \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$ *is well-defined.*

Proof. Let $\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{g_n}{t_n}\right]$ then $f_n \bullet t_m = g_m \bullet \gamma_n$. Employing the Hartley transform and using Eq. (10) we get $Hf_n \curlyvee Ht_m = Hg_m \curlyvee H\gamma_n$. Therefore

$$\left[\frac{Hf_n}{H\gamma_n}\right] = \left[\frac{Hg_n}{Ht_n}\right].$$

The theorem is completely proved.

Theorem 5.2. $\boldsymbol{\kappa}$:$\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$ *is one to one.*

Proof. Let $\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right] = \boldsymbol{\kappa}\left[\frac{g_n}{t_n}\right]$ then $Hf_n \curlyvee Ht_m = Hg_m \curlyvee H\gamma_n$. Using the fact that the classical Hartley transform is one to one and upon employing Eq. (10) we get $f_n \bullet t_m = g_m \bullet \gamma_n$. Therefore $\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{g_n}{t_n}\right]$. Hence the theorem.

Theorem 5.3. $\boldsymbol{\kappa}$:$\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$ *is continuous with respect to* δ *convergence.*

Proof. Let $x_n \to 0$ in $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ as $n \to \infty$, then using [7], $x_n = \left[\frac{f_{n,i}}{\gamma_i}\right]$ for some $f_{n,i}$ where $f_{n,i} \to 0$ as $n \to \infty$. Applying the Hartley transform yields $Hf_{n,i} \to 0$ as $n \to \infty$. Thus $\boldsymbol{\kappa}x_n \to 0$ as $n \to \infty$. This completes the proof.

Theorem 5.4. $\boldsymbol{\kappa}$:$\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$ *is linear .*

Proof of this theorem is straightforward.

Definition 5.5. *Let* $\left[\frac{Hf_n}{H\gamma_n}\right] \in \mathbb{M}_H$ *then we define the inverse generalized Hartley transform to be the mapping*

$$\boldsymbol{\kappa}^{-1}\left[\frac{Hf_n}{H\gamma_n}\right] = \left[\frac{f_n}{\gamma_n}\right] \tag{15}$$

in $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$.

Theorem 5.6. $\boldsymbol{\kappa}^{-1} : \mathbb{M}_H \to \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ *is a well-defined and linear mapping.*

The proof is analoguous to that of Theorems 5.1 and 5.4, and thus avoided.

Theorem 5.7. $\boldsymbol{\kappa}^{-1} : \mathbb{M}_H \to \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ *is one to one.*

Theorem 5.8. $\boldsymbol{\kappa}^{-1} : \mathbb{M}_H \to \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ *is continuous with respect to $\delta-$convergence.*

The proof of Theorems 5.7 and 5.8 are analoguous to that of Theorem 5.2 and Theorem 5.3, respectively. Details are avoided.

Theorem 5.9. *The mapping $\boldsymbol{\kappa}$:$\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$ is surjective.*

Proof. Let $\left[\frac{Hf_n}{H\gamma_n}\right] \in \mathbb{M}_H$ be arbitrary, then $Hf_n \curlyvee H\gamma_m = Hf_m \curlyvee H\gamma_n$ for every $m, n \in \mathbf{N}$. Using Eq. (10), $H(f_n \bullet \gamma_m) = H(f_m \bullet \gamma_n)$, for every $m, n \in \mathbf{N}$. Hence the Boehmian $\left[\frac{f_n}{\gamma_n}\right] \in \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ satisfies the equation $\boldsymbol{\kappa}\left[\frac{f_n}{\gamma_n}\right] = \left[\frac{Hf_n}{H\gamma_n}\right]$. This complete the proof of the lemma.

Theorem 5.10. $\boldsymbol{\kappa}$:$\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet) \to \mathbb{M}_H$, $\boldsymbol{\kappa}^{-1} : \mathbb{M}_H \to \mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ *are continuous with respect to Δ convergence.*

Proof. Let $x_n \xrightarrow{\Delta} x$ in $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$ as $n \to \infty$. Then, there is $f_n \in \mathbb{E}(\mathbb{R})$ and $(\gamma_n) \in \Delta$ such that $(x_n - x) \bullet \gamma_n = \left[\frac{f_n \bullet \gamma_n}{\gamma_n}\right]$ and $f_n \to 0$ as $n \to \infty$. Employing the Hartley transform implies $\boldsymbol{\kappa}((x_n - x) \bullet \gamma_n) = \left[\frac{H(f_n \bullet \gamma_n)}{H\gamma_n}\right]$. Hence $\boldsymbol{\kappa}((x_n - x) \bullet \gamma_n) = \left[\frac{Hf_n \curlyvee H\gamma_n}{H\gamma_n}\right] \sim Hf_n \to 0$ as $n \to \infty$ in $\mathbb{M}_H$. Therefore $\boldsymbol{\kappa}x_n \to \boldsymbol{\kappa}x$ as $n \to \infty$. Next, let $y_n \xrightarrow{\Delta} y$ in $\mathbb{M}_H$ as $n \to \infty$, then we find $F_k \in \mathbb{E}(\mathbb{R})$ such that $(y_n - y) \curlyvee \gamma_n = \left[\frac{F_n \curlyvee H\gamma_n}{H\gamma_n}\right]$ and $F_n \to 0$ as $n \to \infty$ for some $(\gamma_n) \in \Delta$ and $F_n = Hf_n$.

Next, applying Eq. (10),

$$\boldsymbol{\kappa}^{-1}((y_n - y) \curlyvee H\gamma_n) = \left[\frac{H^{-1}(F_n \curlyvee H\gamma_n)}{\gamma_n}\right].$$

Thus $\boldsymbol{\kappa}^{-1}((y_n - y) \curlyvee H\gamma_n) = \left[\frac{f_n \bullet \gamma_n}{\gamma_n}\right] \sim f_n \to 0$ as $n \to \infty$ in $\mathbb{E}(\mathbb{R})$. Thus $\boldsymbol{\kappa}^{-1}((y_n - y) \curlyvee H\gamma_n) = (\boldsymbol{\kappa}^{-1}y_n - \boldsymbol{\kappa}^{-1}y) \bullet \gamma_n \to 0$ as $n \to \infty$. Hence, $\boldsymbol{\kappa}^{-1}y_n \xrightarrow{\Delta} \boldsymbol{\kappa}^{-1}y$ as $n \to \infty$ in $\mathbb{M}(\mathbb{E}, \mathbb{D}, \Delta, \bullet)$. This completes the proof of the theorem.

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