UPPER TOPOLOGICAL GENERALIZED GROUPS

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Abstract. Here, we introduce the notion of generalized universal covers for topological generalized groups and present a method for constructing new topological generalized groups by using of universal covers. As a result a generalization of notion of fundamental groups which is called the generalized fundamental groups is deduced.

Key words and phrases: universal cover, topological generalized group, fundamental group, semilocally simple connected space, locally compact topological group.

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1. Introduction

A new mathematics, isomathematics, was proposed by Santilli when he was studying the mathematical models for electroweak and gravitational theories [8]. The notion of generalized groups, first was introduced by Molaei [4], [5], has an important role in the construction of a geometric unified theory by use of Santilli's isotheory.

Molaei used generalized groups in order to introduce a new kind of dynamics on top spaces [4]. He showed that each generalized group is isometric to a Rees matrix semigroup, see [4]. Also, he introduced the notion of topological generalized groups and proved that if X and Y are Hausdroff topological spaces, G is a topological group and $s : Y \times X \to G$ is a continuous mapping, then the Rees matrix $P = X \times G \times Y$ is a topological generalized group [4]. Topological generalized groups can also be used for modelizing the set of genetic codes, for more details see [5]. Recently, Farhangdoost and Molaei presented a method for constructing new top spaces by using of universal covering spaces of special Lie subsemigroups of a top space, see [6]. Moreover, they deduced a generalization of the notion of fundamental groups which was a completely simple semigroup. Here, we extend their results for semilocally simply connected topological generalized groups. Moreover, we use a generalized notion of universal cover which is developed by Berestovskii and Plaut in the covering group theory for a category of coverable topological groups which requires any form of local simple connectivity [1], [2]. Then, by using of this universal cover for a locally arcwise connected, locally compact topological generalized group, we construct a new topological generalized group.

2. Preliminaries and main results

We introduce some common notations and preliminaries. First, we recall the definition of a generalized group. A generalized group is a non-empty set G admitting an operation called multiplication with the following properties:

i) $(xy)z = x(yz)$, for all $x, y, z \in G$;

ii) for each x belongs to G, there exists a unique element in G, we denote by $e(x)$, such that $x \cdot e(x) = e(x) \cdot x = x$;

iii) for each $x \in G$, there exists $y \in G$ such that $xy = yx = e(x)$.

One can see that each x in a generalized group G has a unique inverse in G [4], we denote it by x^{-1}.

A topological generalized group is a generalized group G equipped with a Hausdorff topology such that the mappings $m : G \times G \to G$, defined by $(g, h) \mapsto g \cdot h$ and $m' : G \to G$, defined by $g \mapsto g^{-1}$ are continuous.

A topological generalized group G is called a normal topological generalized group if G is a normal generalized group, i.e., $e(xy) = e(x) \cdot e(y)$, for all $x, y \in G$.

Now, let G be a normal topological generalized group and let

$$G_{e(g)} = \{h \in G : e(g) = e(h)\}, \text{ for each } g \in G. \text{ Then, } G = \bigcup_{g \in G} G_{e(g)}.$$

It is easy to see that, for each $g \in G$, $G_{e(g)}$ with subspace topology and product of G is a topological group.

We note that if G and H are two normal topological generalized groups and $f : G \to H$ is an algebraic homomorphism, then $f(e(g)) = e(f(g))$ and $f : G_{e(g)} \to H_{e(f(g))}$ is a group homomorphism, for each $g \in G$.

Let G be a topological space. G is called semilocally simply connected, if for each $x \in G$, there is an open set U of x such that the inclusion of U in G induces the trivial homomorphism on their fundamental groups. Topological space G is called simply connected if G is arcwice connected and $\pi_1(G) \simeq \{e\}$, where $\pi_1(G)$ denotes the (Poincare) fundamental group of G.

If X is a topological space and if $\mathcal{C}$ is a collection of subspaces of X whose union is X, the topology of X is said to be coherent with the collection $\mathcal{C}$, provided a set A is closed in X if and only if $A \cap C$ is closed in C, for each $C \in \mathcal{C}$. It is equivalent to require that U is open in X if and only if $U \cap C$ is open in C, for each $C \in \mathcal{C}$.

Let G and H be two topological groups and $\phi : G \to H$ be an open epimorphism with discrete kernel. Then ϕ is called a (traditional) cover. It is easy to see that, if ϕ is a cover then ϕ is a local homeomorphism. A universal cover for a topological group H is a covering epimorphism $\phi : G \to H$ such that for any cover $\rho : F \to H$ of topological groups, there is a homomorphism $\psi : G \to F$ such that $\phi = \rho\psi$. If G and F are connected, it follows easily that ψ is a cover and unique, see [1].

In [1], Berestovskii and Plaut utilized a generalized notion of cover, namely an open epimorphism between topological groups whose kernel is central and prodiscrete, i.e., the inverse limit of discrete groups. They proved that, for a large category C of topological groups, called coverable topological groups, the following assertions hold:

(1) For every $G \in \mathcal{C}$, there exists a cover $\phi : \widetilde{G} \to G$.

(2) Covers are morphisms in $\mathcal{C}$, (i.e., the composition of covers between elements of $\mathcal{C}$ is a cover).

(3) The cover $\phi : \widetilde{G} \to G$ has the traditional universal property of the universal cover in the category $\mathcal{C}$ with covers as morphisms.

Here, we define the notion of a cover for topological generalized groups. Let G and $\widetilde{G}$ be two normal topological generalized groups and $\phi : \widetilde{G} \to G$ be an algebraic homomorphism. So, the restriction of ϕ to each $\widetilde{G}_{e(\widetilde{g})}$ is a group homomorphism from $\widetilde{G}_{e(\widetilde{g})}$ to $G_{e(\phi(\widetilde{g}))}$. For simplicity, we denote it by $\phi_{\widetilde{g}}$. We say that ϕ is a (traditional) generalized cover of topological generalized groups, if ϕ is an open epimorphism with discrete kernel, where $\ker\phi = \bigcup_{\widetilde{g} \in \widetilde{G}} \ker\phi_{\widetilde{g}}$.

Moreover, by using of notion of covers in the sense of Berestovskii and Plaut, we define another notion of cover for topological generalized groups. We say that an open epimorphism $\phi : \widetilde{G} \to G$ of normal topological generalized groups $\widetilde{G}$ and G is a generalized cover in the sense of Berestovskii and Plaut, if the restriction of ϕ to each $\widetilde{G}_{e(\widetilde{g})}$ is an open epimorphism of topological groups with central and prodiscrete kernel.

Let G be a normal topological generalized group and $\phi : \widetilde{G} \to G$ be a universal generalized cover of G. Then we call the pair $(\widetilde{G}, \phi)$ an upper topological generalized group.

In this article, we present a method for constructing new topological generalized groups by using of two kinds of universal covers, traditional universal covers and the universal covers in the sense of Berestovskii and Plaut. As a result, generalization of th notion of fundamental groups, which are generalized fundamental groups is deduced.

Theorem 2.1. *Let G be a locally acrwise connected and semilocally simply connected normal topological generalized group such that its topology coherent with the collection $\{G_{e(g)} : g \in G\}$. Then there exist a normal topological generalized group $\widetilde{G}$ and a (traditional) generalized universal cover $\phi : \widetilde{G} \to G$ associated to G. Moreover, $\widetilde{G}_{e(\widetilde{g})}$ is a connected and simply connected topological group for each $\widetilde{g} \in \widetilde{G}$.*

We denote the restriction of ϕ to $\widetilde{G}_{e(\widetilde{g})}$ by $\phi_g : \widetilde{G}_{e(\widetilde{g})} \to G_{e(g)}$.

Corollary 2.2. *With the above assumption,* $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ *is a discrete topological generalized subgroup of $\widetilde{G}$. In particular, $\pi_1(G_{e(g)})$ is abstractly isomorphic to* $\ker\phi_g$*, where $\pi_1(G_{e(g)})$ is the (Poincaré) fundamental group of $G_{e(g)}$.*

Also, we have the following results for locally compact topological generalized groups. In this case, the notion of (traditional) generalized universal cover substituted by generalized universal cover in the sense of Berestovskii and Plaut.

Theorem 2.3. *Let G be a locally arcwise connected, locally compact normal generalized topological group such that its topology is coherent with the collection $\{G_{e(g)} : g \in G\}$. Then there exist a normal topological generalized group $\widetilde{G}$ and a generalized universal cover (in the sense of Berestovskii and Plaut) $\phi : \widetilde{G} \to G$ associated to G. Moreover, $\widetilde{G}_{e(\widetilde{g})}$ is connected and simply connected topological group for each $\widetilde{g} \in \widetilde{G}$.*

Corollary 2.4. *With the above assumptions,* $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ *is a prodiscrete topological generalized subgroup of $\widetilde{G}$. In particular, $\pi_1(G_{e(g)})$ is abstractly isomorphic to* $\ker\phi_g$.

3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1 and Corollary 2.2. Let G be a topological group. If G is connected, locally arcwise connected and semilocally simply connected, then G has a universal cover $\phi : \widetilde{G} \to G$. Moreover, $\widetilde{G}$ is a connected, locally arcwise connected and simply connected topological group. In fact, we have the following results (see [9] and [1]):

Proposition 3.1. *Let G be connected and locally arcwise connected. G admits a universal cover $\phi : \widetilde{G} \to G$ if and only if it is semilocally simply connected. Moreover, ϕ is unique up to isomorphism.*

Also, we need the following proposition of [1], (Proposition 81).

Proposition 3.2. *Let $G, \widetilde{G}$ be topological groups and $\phi : \widetilde{G} \to G$ be a cover. Suppose that X is a connected, locally arcwise connected and simply connected* *topological space. If $f:(X,p)\to(G,e)$ is a continuous function, then there is a unique lift $g:(X,p)\to(\widetilde{G},\widetilde{e})$ such that $f=\phi\circ g$.*

Proposition 3.3. *Assume that G is a locally arcwise connected and semilocally simply connected topological generalized group such that its topology coherent with $\{G_{e(g)}:g\in G\}$. Then each $G_{e(g)}$ has a universal cover, $\phi_g:\widetilde{G}_g\to G_{e(g)}$, which is unique up to isomorphism.*

Proof. Let G be a normal topological generalized group. Then the mapping $g\mapsto e(g)$ is continuous, see [4]. Therefore, $G_{e(g)}$ is a closed subspace of G. If we put $G=\bigcup_{g\in G}G_{e(g)}$, then for each $g,h\in G$, the topological groups $G_{e(g)}$ and $G_{e(h)}$ are either disjoint or identical, i.e., $G_{e(g)}=G_{e(h)}$.

Suppose that the topology of G is coherent with the collection $\{G_{e(g)}:g\in G\}$. Then, each $G_{e(g)}$ is also an open subspace of G. Moreover, if G is locally arcwise connected and semilocally simply connected, then each $G_{e(g)}$ is an arcwise connected, locally arcwise connected and semilocally simply connected topological group.

Now, by Proposition 3.1, $G_{e(g)}$ has a universal cover, say $\phi_g:\widetilde{G}_g\to G_{e(g)}$, which is unique up to isomorphism. ∎

Note that, by Proposition 3.1, $\widetilde{G}_g$ is connected and simply connected topological group. Also, $\widetilde{G}_g$ is locally arcwise connected. This implies that $\widetilde{G}_g\times\widetilde{G}_g$ is also connected, locally arcwise connected and simply connected topological space. Therefore, by Proposition 3.2, the mapping $m\circ(\phi_g\times\phi_g):\widetilde{G}_g\times\widetilde{G}_g\to G_{e(g)}$, has a unique lifting $\widetilde{m}:\widetilde{G}_g\times\widetilde{G}_g\to\widetilde{G}_g$ such that $\widetilde{m}(\widetilde{e}_g,\widetilde{e}_g)=\widetilde{e}_g$, where $\widetilde{e}_g$ is the unit element of $\widetilde{G}_g$.

Proposition 3.4. *$\widetilde{G}_g$ with the multiplication defined by $\widetilde{m}$ is also a topological group. Moreover, its structure group with product $\widetilde{m}$ is the same as to its original structure group up to isomorphism.*

Proof. First, we have

$$\begin{aligned}\phi_g\circ(\widetilde{m}\circ(id_{\widetilde{G}_g}\times\widetilde{m})) &= m\circ(\phi_g\times\phi_g)\circ(id_{\widetilde{G}_g}\times\widetilde{m})\\ &= m\circ(\phi_g\circ\widetilde{m}\times\phi_g)\\ &= m\circ((m\circ(\phi_g\times\phi_g))\times\phi_g)\\ &= m\circ(m\times id_{G_{e(g)}})\circ(\phi_g\times\phi_g\times\phi_g)\end{aligned}$$

and

$$\begin{aligned}\phi_g\circ(\widetilde{m}\circ(\widetilde{m}\times id_{\widetilde{G}_g})) &= m\circ(\phi_g\times\phi_g)\circ(\widetilde{m}\times id_{\widetilde{G}_g})\\ &= m\circ(\phi_g\circ\widetilde{m}\times\phi_g)\\ &= m\circ((m\circ(\phi_g\times\phi_g))\times\phi_g)\\ &= m\circ(m\times id_{G_{e(g)}})\circ(\phi_g\times\phi_g\times\phi_g).\end{aligned}$$

Since the multiplication on G is associative, it follows that $\widetilde{m} \circ (id_{\widetilde{G}_g} \times \widetilde{m})$ and $\widetilde{m} \circ (\widetilde{m} \times id_{\widetilde{G}_g})$ are the lifts of the space map from $\widetilde{G}_g \times \widetilde{G}_g \times \widetilde{G}_g$ into $G_{e(g)}$. Since both maps map $(\widetilde{e}_g, \widetilde{e}_g, \widetilde{e}_g)$ into $\widetilde{e}_g$, it follows that they are identical, i.e., the operation $\widetilde{m}$ is associative.

Also, we have

$$\phi_g(\widetilde{m}(\widetilde{g}, \widetilde{e}_g)) = m(\phi_g(\widetilde{g}, e(g))) = \phi_g(\widetilde{g}).$$

Therefore, $\widetilde{g} \mapsto \widetilde{m}(\widetilde{g}, \widetilde{e}_g)$ is the lifting of $\phi_g : \widetilde{G}_g \to G_{e(g)}$. Since $\widetilde{m}(\widetilde{e}_g, \widetilde{g}) = \widetilde{e}_g$, this map is the identity on $\widetilde{G}_g$, i.e., $\widetilde{m}(\widetilde{g}, \widetilde{e}_g) = \widetilde{g}$, for all $\widetilde{g} \in \widetilde{G}_g$.

Analogously, we have

$$\phi_g(\widetilde{m}(\widetilde{e}_g, \widetilde{g})) = m(e(g); \phi_g(\widetilde{g})) = \phi_g(\widetilde{g}).$$

Hence, $\widetilde{g} \mapsto \widetilde{m}(\widetilde{e}_g, \widetilde{g})$ is the lifting of $\phi_g : \widetilde{G}_g \to G_{e(g)}$. Since $\widetilde{m}(\widetilde{e}_g, \widetilde{g}) = \widetilde{e}_g$, this map is the identity on $\widetilde{G}_g$, i.e., $\widetilde{m}(\widetilde{e}_g, \widetilde{g}) = \widetilde{g}$ for all $\widetilde{g} \in \widetilde{G}_g$.

It follows that $\widetilde{e}_g$ is the identity in $\widetilde{G}_g$. Let $\widetilde{m}' : \widetilde{G}_g \to \widetilde{G}_g$ be the lifting of the map $m' \circ \phi_g : \widetilde{G}_g \to \widetilde{G}_{e(g)}$ such that $\widetilde{m}'(\widetilde{e}_g) = \widetilde{e}_g$.

Then we have

$$\begin{aligned} \phi_g(\widetilde{m}(\widetilde{g}, \widetilde{m}'(\widetilde{g}))) &= m(\phi_g(\widetilde{g}); \phi_g(\widetilde{m}'(\widetilde{g}))) \\ &= m(\phi_g(\widetilde{g}), (\phi_g(\widetilde{g}))^{-1}) \\ &= e(g). \end{aligned}$$

Therefore, $\widetilde{g} \mapsto \widetilde{m}(\widetilde{g}, \widetilde{m}'(\widetilde{g}))$ is the lifting of constant map of $\widetilde{G}_g$ into $e(g)$.

Since $\widetilde{m}(\widetilde{e}_g, \widetilde{m}'(\widetilde{e}_g) = \widetilde{e}_g$, we conclude that this map is constant and its value is equal to $\widetilde{e}_g$.

Therefore, we have

$$\widetilde{m}(\widetilde{g}, \widetilde{m}'(\widetilde{g})) = \widetilde{e}_g, \text{ for all } \widetilde{g} \in \widetilde{g}_g.$$

Analogously, we have

$$\begin{aligned} \phi_g(\widetilde{m}'(\widetilde{g}), \widetilde{g}) &= m(\phi_g(\widetilde{m}'(\widetilde{g})), \phi_g(\widetilde{g})) \\ &= m((\phi_g(\widetilde{g})))^{-1} \\ &= e(g). \end{aligned}$$

Therefore, $\widetilde{g} \mapsto \widetilde{m}(\widetilde{m}'(\widetilde{g}), \widetilde{g})$ is the lifting of the constant map of $\widetilde{G}_g$ into $e(g) \in G$. Since $\widetilde{m}(\widetilde{m}'(\widetilde{e}_g), \widetilde{e}_g) = \widetilde{e}_g$, we conclude that this map is constant and its value is equal to $\widetilde{e}_g$.

Therefore, we have $\widetilde{m}(\widetilde{m}'(\widetilde{g}), \widetilde{g}) = \widetilde{g}$, for all $\widetilde{g} \in \widetilde{G}_g$. This implies that any element $\widetilde{g} \in \widetilde{G}_g$ has an inverse $\widetilde{m}'(g) = \widetilde{g}'1$. Therefore, $\widetilde{G}_g$ with operation $\widetilde{m}$ is a group. Moreover, since $\widetilde{m}$ and $\widetilde{m}'$ are continuous, $\widetilde{G}_g$ with this operation is a topological group, which we denote it by $\overline{G}_g$. It is easy to see that $\phi_g : \overline{G}_g \to G_{e(g)}$ is a cover and $\overline{G}$ is a connected and simply connected topological group. Then, by

Proposition 3.1, since $G_{e(g)}$ is semilocally simply connected, the topological groups $\widetilde{G}_g$ and $\overline{G}_g$ are the same up to isomorphism (from now on we use the notation $\widetilde{G}_g$ for both of them). ∎

Now, we construct a new normal topological generalized groups $\widetilde{G}$ such as follows:

Let $\widetilde{G}$ be the disjoint union of $\widetilde{G}_g$, where $g \in G$. We consider a topology on $\widetilde{G}$ which is coherent with the collection $\mathcal{C} = \{\widehat{G}_g : g \in G\}$, provided a set U is open in $\widetilde{G}$ if and only if $U \cap \widetilde{G}_g$ is open in $\widetilde{G}_g$, for each $\widetilde{G}_g \in \mathcal{C}$. Clearly, the topology of $\widetilde{G}_g$ as a subspace of $\widetilde{G}$ is equivalent to original topology of $\widetilde{G}_g$, see [6]. So, $\widetilde{G}_g$ is connected, locally path connected and simply connected as a subspace of $\widetilde{G}$. This implies that $\widetilde{G}_g \times \widetilde{G}_h$ is also connected, locally arcwise connected and simply connected. Then, by Proposition 3.2, the mapping $m\circ(\phi_g \times \phi_h) : \widetilde{G}_g \times \widetilde{G}_h \to G_{e(gh)}$ has a unique lifting $\widetilde{m}_{gh} \times \widetilde{G}_h \to \widetilde{G}_{gh}$ such that $\widetilde{m}_{gh}(\widetilde{e}_g, \widetilde{e}_h) = \widetilde{e}_{gh}$. In this way, we can define the product $\widetilde{m}$ on $\widetilde{G} \times \widetilde{G}$ by $\widetilde{m}(\widetilde{g}, \widetilde{h}) = \widetilde{m}_{gh}(\widetilde{g}, \widetilde{h})$.

Proposition 3.5. $(\widetilde{g}, \widetilde{m})$ *is a normal topological generalized group.*

Proof. First, we show that $\widetilde{m}$ is associative. We have

$$\begin{aligned}
\phi_{g(hk)} \circ (\widetilde{m}_{g(hk)} \circ (id_{\widetilde{G}} \times \widetilde{m}_{hk})) &= m \circ (\phi_g \times \phi_{hk}) \circ (id_{\widetilde{G}} \times \widetilde{m}_{hk}) \\
&= m \circ (\phi_{gh} \circ \widetilde{m} \times \phi_k) \\
&= m \circ (m \circ (m \circ \phi_g \times \phi_h) \times \phi_k) \\
&= m \circ (m \times id_G) \circ (\phi_g \times \phi_h \times \phi_k)
\end{aligned}$$

and

$$\begin{aligned}
\phi_{ghk} \circ (\widetilde{m} \circ (\widetilde{m} \times id_{\widetilde{G}})) &= m \circ (\phi_{gh} \times \phi_k) \circ (\widetilde{m} \times id_{\widetilde{G}}) \\
&= m \circ (\phi_{ghk} \circ \widetilde{m} \times \phi_k) \\
&= m \circ ((m \circ (\phi_g \times \phi_h)) \times \phi_k) \\
&= m \circ (m \times id_G) \circ (\phi_g \times \phi_h \times \phi_k).
\end{aligned}$$

Since the multiplication on G is associative, it follows that $\widetilde{m} \circ (id_{\widetilde{G}} \times \widetilde{m})$ and $\widetilde{m} \circ (\widetilde{m} \times id_{\widetilde{G}})$ are the lifts of the same map from $\widetilde{G}_g \times \widetilde{G}_h \times \widetilde{G}_k$ into $G_{e(ghk)}$. Since both maps map $(\widetilde{e}_g, \widetilde{e}_h, \widetilde{e}_k)$ into $\widetilde{e}_{e(ghk)}$, it follows that they are identical, i.e., the operation $\widetilde{m}$ is associative. Also, for each $\widetilde{h} \in \widetilde{G}$, there exist $g \in G$ such that $\widetilde{h} \in \widetilde{G}_g$. Therefore, $\widetilde{m}(\widetilde{h}, \widetilde{e}_g) = \widetilde{m}(\widetilde{e}_g, \widetilde{h}) = \widetilde{h}$ and $\widetilde{h}$ has a unique inverse in $\widetilde{G}_g$. (Note that $\widetilde{G}_g$'s are disjoint.) Clearly, the product $\widetilde{m}$ and the mapping $\widetilde{m}' : \widetilde{G} \to \widetilde{G}$, $\widetilde{g} \mapsto \widetilde{g}^{-1}$ are continuous and this implies that $\widetilde{G}$ with product $\widetilde{m}$ is a normal topological generalized group. ∎

We note that, the mapping $\phi : \widetilde{G} \to G$ defined by $\phi(\widetilde{g}) = \phi_g(\widetilde{g})$, where $\widetilde{g} \in \widetilde{G}$ for some $g \in G$, is a homomorphism of topological generalized groups $\widetilde{G}$ and G.

If we define the kernel of ϕ by $\ker \phi = \bigcup_{g \in G} \ker \phi_g$, then $\ker \phi$ is discrete. Moreover, ϕ is an open mapping. For, let $\widetilde{U} \subset \widetilde{G}$ be open in $\widetilde{G}$. Then, since the topology of $\widetilde{G}$ is coherent with the collection $\{\widetilde{G}_g : g \in G\}$, $\widetilde{U} \cap \widetilde{G}_g$ is open in $\widetilde{G}_g$, for each $g \in G$. On the other hand, ϕ_g is an open mapping. Therefore, $\phi_g(\widetilde{U} \cap \widetilde{G})$ is open in $G_{e(g)}$. Since $\phi(\widetilde{U}) = \bigcup_{g \in G} \phi_g(\widetilde{U} \cap \widetilde{G}_g)$, this implies that $\phi(\widetilde{U})$ is open in G, i.e., ϕ is an open mapping. Hence, ϕ is an open epimorphism with discrete kernel and restriction of ϕ to each $\widetilde{G}_g$ is a universal cover of $\widetilde{G}_g$ to $G_{e(g)}$. Therefore, ϕ is a universal generalized cover between topological generalized groups $\widetilde{G}$ and G. This complete the proof of Theorem 2.1. ∎

In the sequel, we need the following result of [1] (see Corollary 85).

Proposition 3.6. *If G is a topological group, $\phi : \widetilde{G} \to G$ is a universal cover, and $\widetilde{G}$ is arcwise connected and $\pi_1(\widetilde{G}) = e$, then $\pi_1(G)$ is absolutely isomorphic to* $\ker \phi$.

Now, we consider the universal covers $\phi_g : \widetilde{G}_g \to G_{e(g)}$, for each $g \in G$. As we have already seen that, $\widetilde{G}_g$ is connected, locally arcwise connected and simply connected. So, $\widetilde{G}_g$ is also arcwise connected and Proposition 3.2 implies that $\pi_1(G_{e(g)})$ is absolutely isomorphic to $\ker \phi_g$. Then, $\ker \phi = \bigcup_{g \in G} \ker \phi_g \simeq \bigcup_{g \in G} \pi_1(G_{e(g)})$. It is easy to see that $\ker \phi$ is also a topological generalized subgroup of $\widetilde{G}$. Therefore, the assertion of Corollary 2.4 holds.

Proposition 3.7. *Let G be a locally arcwise connected and semilocally simply connected normal topological generalized group such that its topology is coherent with the collection $\{G_{e(g)} : g \in G_g\}$. If $(\widetilde{G}, \phi)$ and $(\widetilde{G}, \psi)$ be two upper topological generalized groups of G, then* $\ker \phi$ *is isomorphic to* $\ker \psi$.

Proof. Since the topology of G is coherent with $\{G_{e(g)} : g \in G_g\}$, then $G_{e(g)}$ is open in G, for each $g \in G$. This implies that $G_{e(g)}$ is semilocally simply connected, connected and locally arcwise connected. Therefore, by Proposition 3.2, the universal cover $(\widetilde{G}_g, \phi_g)$ is unique up to isomorphism. Then, $\ker \phi_g$ is isomorphic to $\ker \psi_g$ and this implies that $\ker \phi$ is also isomorphic to $\ker \psi$. ∎

4. Locally compact topological generalized groups

In this section, we consider the locally compact topological generalized groups and we use a generalization notion of universal cover which is developed by Berestivskii and Plaut, that is $\phi : \widetilde{G} \to G$ is a cover of topological groups G and $\widetilde{G}$ if ϕ is an open epimorphism whose kernel is central and prodiscrete (i.e., the inverse limit of discrete groups), see [2]. They proved that, for any topological group G there is a topological group $\widetilde{G}$ and a natural homomorphism $\phi : \widetilde{G} \to G$. In particular, if G is coverable then $\widetilde{G}$ is coverable and ϕ is a universal cover in the category of coverable groups and covers. In the sequel, we use the following results of [2].

Theorem 4.1. *Let G be a locally compact topological group. Then the following are equivalent:*

1. *G is coverable,*
2. *$\phi : \widetilde{G} \to G$ is a cover,*
3. *$\phi : \widetilde{G} \to G$ is open and G is connected,*
4. *$\phi : \widetilde{G} \to G$ is surjective,*
5. *G is connected and locally arcwise connected.*

Moreover, if G is metrizable, then G is coverable if and only if G is connected and locally connected.

Proof of Theorem 2.3. By Theorem 4.1, if G is connected, locally arcwise connected and locally compact topological group then G is coverable and natural homomorphism $\phi : \widetilde{G} \to G$ is a cover.

Moreover, $\widetilde{G}$ is simply connected, see [1]. Therefore, if G is connected, locally arcwise connected and locally compact topological group, then by Proposition 3.2, the mapping $m \circ (\phi \times \phi) : \widetilde{G} \times \widetilde{G} \to G$ has a unique lifting $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that $\widetilde{m}(\widetilde{e}, \widetilde{e}) = \widetilde{e}$, where $\widetilde{e}$ is the unit element of $\widetilde{G}$. Now, Proposition 3.4 implies that $\widetilde{G}$ with product $\widetilde{m}$ is a topological group. Moreover, its structure group with product $\widetilde{m}$ is the same as its original structure group up to isomorphism (we note that the Proposition 3.2 also holds for covers in the sense of Berestovskii and Plaut).

Now, suppose that G be a locally arcwise connected and locally compact topological generalized group with its topology coherent with the collection $\{G_{e(g)} : g \in G\}$. Then, each $G_{e(g)}$ is connected, locally arcwise connected and locally compact topological group and therefore has a natural cover $\phi_g : \widetilde{G}_g \to G_{e(g)}$.

Now, we construct a new normal topological generalized group $\widetilde{G}$ as follows:

Let $\widetilde{G}$ be the disjoint union of $\widetilde{G}_g$, where $g \in G$. We consider a topology on $\widetilde{G}$ which is coherent with the collection $\mathcal{C} = \{\widetilde{G}_g : g \in G_g\}$. Therefore, the topology of $\widetilde{G}_g$ as a subspace of $\widetilde{G}$ is equivalent to original topology of $\widetilde{G}_g$. Since ϕ_g is surjective and $G_{e(g)}$ is connected, then $\widetilde{G}_g$ is also connected. Moreover, Theorem 3 of [2] implies that $\widetilde{G}_g$ is locally arcwise connected. We have already seen that $\widetilde{G}_g$ is simply connected. Therefore, $\widetilde{G}_g \times \widetilde{G}_h$ is also connected, locally arcwise connected and simply connected. Then, by Proposition 3.2, the mapping $m \circ (\phi_g \times \phi_h) : \widetilde{G}_g \times \widetilde{G}_h \to G_{e(gh)}$ has a unique lifting $\widetilde{m}_{gh} : \widetilde{G}_g \times \widetilde{G}_h \to \widetilde{G}_{gh}$ such that $\widetilde{m}_{gh}(\widetilde{e}_g, \widetilde{e}_h) = \widetilde{e}_{gh}$. In this way, we can define the product $\widetilde{m}$ on $\widetilde{G} \times \widetilde{G}$ by $\widetilde{m}(e_g, e_h) = \widetilde{m}_{gh}(\widetilde{g}, \widetilde{h})$, where $\widetilde{g} \in \widetilde{G}_g$ and $\widetilde{h} \in \widetilde{G}_h$. Now, Proposition 3.5 implies that $(\widetilde{G}, \widetilde{m})$ is a normal topological generalized group. Clearly, the mapping $\phi : \widetilde{G} \to G$ defined by $\phi(\widetilde{g}) = \phi_g(\widetilde{g})$, where $\widetilde{g} \in \widetilde{G}_g$ for some $g \in G$ is an algebraic homomorphism of topological generalized groups $\widetilde{G}$ and G. We define the kernel of ϕ by $\ker\phi = \bigcup_{g\in G} \ker\phi_g$. Moreover, ϕ is an open mapping (for proof, see the argument used in proof of Theorem 2.1). Hence, ϕ is a generalized cover. This complete the proof of Theorem 4.1. ∎

Proposition 4.2. *The kernel of* ϕ, $\ker\phi$, *is totally disconnected.*

Proof. By Lemma 32 of [1], prodiscrete topological groups are totally disconnected. So, for each $g \in G$, $\ker\phi_g$ is totally disconnected. Since $\ker\phi = \bigcup_{g\in G} \ker\phi_g$ and topology of $\widetilde{G}$ is coherent with $\{\widetilde{G}_g : g \in G\}$, then $\ker\phi$ is also totally disconnected. ∎

Now, we need the following results of [2].

Proposition 4.3. *If G is locally compact topological group, then $\pi_1(G)$ is abstractly isomorphic to the prodiscrete topological group* $\ker\phi$*, where $\phi : \widetilde{G} \to G$ is the natural homomorphism.*

If we consider the universal covers $\phi_g : \widetilde{G}_g \to G_{e(g)}$, for each $g \in G$, as we have already seen, $G_{e(g)}$ is locally compact topological group and then, by Proposition 4.3, $\pi_1(G)$ absolutely isomorphic to the prodiscrete topological group $\ker\phi_g$. Then, $\ker\phi = \cup \ker\phi_g \simeq \cup\pi_1(G_{e(g)})$, which is also a topological generalized subgroup of $\widetilde{G}$. Therefore, Corollary 2.4 holds.

Theorem 4.4. *Let G be a normal topological generalized group. Then $G_{e(g)}$ and $G_{e(h)}$ are homomorphic, for each $g, h \in G$.*

Proof. By Lemma 2.1 of [4], if G is a topological generalized group, then $e(g)G = gG$ for each $g \in G$. Let $g, h \in G$. Then $e(g)h = gg'$, for some $g' \in G$,

$$e(g)e(h) = e(e(g)h) = e(gg') = e(g)e(g') \Longrightarrow e(h) = e(g').$$

So,

$$gg'h^{-1} = e(g)hh^{-1} = e(g)e(h) = e(g)e(g') = e(gg').$$

Therefore, $(gg')^{-1} = h^{-1}$, that is $gg' = h$.

Now, we define $R_{g'} : G_{e(g)} \to G_{e(h)}$, by right translation, $k \mapsto kg'$. Then mapping $R_{g'}$ is well-defined, since

$$e(kg') = e(k)e(g') = e(g)e(g') = e(gg') = e(h).$$

On the other hand, since product on G is continuous, then $R_{g'}$ is continuous.

Also $(R_{g'})^{-1} = R_{g'-1}$. For,

$$\begin{aligned} R_{g'} \circ R_{g'-1}(k) &= R_{g'}(kg'^{-1}) \\ &= (kg'^{-1})g' \\ &= k(g'^{-1}g') \\ &= ke(g') \\ &= ke(h) \\ &= ke(k) \\ &= k. \end{aligned}$$

Similarly, $R_{g'-1} \circ R_{g'} = id_{G_{e(g)}}$. So, $R_{g'}$ is a homeomorphism and $G_{e(g)}$ is homeomorphic to $G_{e(h)}$. ∎

Remark 4.5. We note that, if G is a normal topological generalized group that satisfies the assumptions of Theorem 2.1 or 2.3, then, for each $g \in G$, $G_{e(g)}$ is a path component of G. Therefore, by the above theorem, path components of G are homeomorphic. This implies that the fundamental group of G does not depend on the base point.

Remark 4.6. We note that Biss [3] puts a topology on the fundamental groups of topological spaces. Let (X, x) be a pointed space. He equipped the space of continuous based maps $Hom((S^1, 1), (X, x))$ with the compact-open topology. Then by using the surjection $Hom((S^1, 1), (X, x)) \to \pi_1(X, x)$, he defined a quotient topology on $\pi_1(X, x)$. As we saw here, by using the notion of universal covers for coverable topological groups in the sense of Berestovskii and Plaut, the fundamental groups admit a natural prodiscrete topology as the kernel of their universal covers. The fundamental groups with this topology are always Hausdorff, however with the compact-open topology introduced by Biss, in general, they would not be a Hausdorff topological space. So, in general, these two topology are not the same.

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