# HOMOMORPHISMS AND EPIMORPHISMS OF SOME HYPERGROUPS

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**Abstract.** By a *homomorphism* of a hypergroup $(H, \circ)$ we mean a function $f : H \to H$ satisfying $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in H$. A homomorphism $f$ of a hypergroup $(H, \circ)$ is called an *epimorphism* if $f(H) = H$. For a hypergroup $(H, \circ)$, denote by $\mathrm{Hom}(H, \circ)$ and $\mathrm{Epi}(H, \circ)$ the set of all homomorphisms and the set of all epimorphisms of $(H, \circ)$, respectively. For a positive integer $n$, let $(\mathbb{Z}, \circ_n)$ be the hypergroup where $x \circ_n y = x + y + n\mathbb{Z}$ for all $x, y \in \mathbb{Z}$. In this paper, we characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Epi}(\mathbb{Z}, \circ_n)$. In addition, we show that $|\mathrm{Hom}(\mathbb{Z}, \circ_n)|=|\mathrm{Epi}(\mathbb{Z}, \circ_n)|=2^{\aleph_0}$.

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## 1. Introduction

A *hyperoperation* on a nonempty set $H$ is a function $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}(H)$ is the power set of $H$ and $\mathcal{P}^*(H) = \mathcal{P}(H) \smallsetminus \{\emptyset\}$. The value of $(x, y) \in H \times H$ under $\circ$ is denoted by $x \circ y$. The system $(H, \circ)$ is called a *hypergroupoid*. For $A, B \subseteq H$ and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

The hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ satisfying

$$H \circ x = x \circ H = H \quad \text{for all } x \in H.$$

Then hypergroups are a generalization of groups.

By a *homomorphism* of a hypergroup $(H, \circ)$ we mean $f : H \to H$ such that

$$f(x \circ y) \subseteq f(x) \circ f(y) \text{ for all } x, y \in H.$$

If the equality is valid, $f$ is called a *good homomorphism* of $(H, \circ)$. A [good] homomorphism of a hypergroup $(H, \circ)$ is called an [*good*] *epimorphism* of $(H, \circ)$ if $f(H) = H$. For a hypergroup $(H, \circ)$, let $\mathrm{Hom}(H, \circ)$ and $\mathrm{Epi}(H, \circ)$ be the set of all homomorphisms and the set of all epimorphisms of $(H, \circ)$, respectively.

If $G$ is a group, $N$ is a normal subgroup of $G$ and $\circ_N$ is the hyperoperation on $G$ defined by

$$x \circ_N y = xyN \quad \text{for all } x, y \in G,$$

then $(G, \circ_N)$ is a hypergroup ([2], p.11). Observe that if $N=\{e\}$, then $(G, \circ_N)=G$ where $e$ is the identity of $G$. Let $\mathbb{Z}$ be the set of integers and $n$ a positive integer. Then $(\mathbb{Z}, \circ_n)$ is a hypergroup where

$$x \circ_n y = x + y + n\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

In [3], the authors characterized the good homomorphisms and good epimorphisms of the hypergroup $(\mathbb{Z}, \circ_n)$. Such homomorphisms were also counted in [3]

The cardinality of a set $X$ is denoted by $|X|$.

For $a \in \mathbb{Z}$, let $g_a : \mathbb{Z} \to \mathbb{Z}$ be defined by $g_a(x) = ax$ for all $x \in \mathbb{Z}$. Then

$$\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{Epi}(\mathbb{Z}, +) = \{g_1, g_{-1}\}.$$

Hence $|\mathrm{Hom}(\mathbb{Z}, +)| = \aleph_0$ and $|\mathrm{Epi}(\mathbb{Z}, +)| = 2$.

Our objective is to

(1) characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ and $\mathrm{Epi}(\mathbb{Z}, \circ_n)$,

(2) show that $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = |\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

The following fact of infinite cardinal numbers will be used. If $p$ is an infinite cardinal number, then $p^p = 2^p$ ([6], p.161).

We note here that some results of homomorphisms and good homomorphisms of certain hypergroups can be seen in [1]. Homomorphisms of some multiplicative hyperrings were also studied in [4] and [5].

Let $\mathbb{Z}^+$ stand for the set of positive integers.

## 2. Homomorphisms of the Hypergroup $(\mathbb{Z}, \circ_n)$

The following lemma is needed to characterize the elements of $\mathrm{Hom}(\mathbb{Z}, \circ_n)$.

**Lemma 2.1.** *Let $G$ be a group, $N$ a normal subgroup of $G$ and $\circ_N$ the hyperoperation on $G$. Then the following statements hold for $f \in \mathrm{Hom}(G, \circ_N)$.*

(i) $f(N) \subseteq N$.

(ii) *For all $x \in G$, $f(xN) \subseteq f(x)N$.*

(iii) *For all* $x, y \in G$, $f(xyN) \subseteq f(xy)N = f(x)f(y)N$.
(iv) *For all* $x \in G$, $f(x^{-1}N) \subseteq f(x^{-1})N = f(x)^{-1}N$.
(v) *For all* $x \in G$ *and* $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N = f(x)^kN$.

**Proof.** First, we recall that for all $x, y \in G$, $xN \cap yN \neq \emptyset$ implies $xN = yN$.

(i) We have that

$$f(N) = f(eeN) = f(e \circ_N e) \subseteq f(e) \circ_N f(e) = f(e)f(e)N.$$

Then $f(e) \in f(N) \subseteq f(e)f(e)N$. Since $G$ is cancellative, we have $e \in f(e)N$ which implies that $N = f(e)N$, so $f(N) \subseteq f(e)f(e)N = N$.

(ii) By (i), $f(e) \in N$. If $x \in G$, then

$$f(xN) = f(xeN) = f(x \circ_N e) \subseteq f(x) \circ_N f(e) = f(x)f(e)N = f(x)N.$$

(iii) Let $x, y \in G$. Then by (ii),

$$f(xyN) \subseteq f(xy)N.$$

We also have that

$$f(xyN) = f(x \circ_N y) \subseteq f(x) \circ_N f(y) = f(x)f(y)N.$$

This implies that $f(xy)N = f(x)f(y)N$. Hence (iii) holds.

(iv) If $x \in G$, then

$$f(N) = f(xx^{-1}N) = f(x \circ_N x^{-1}) \subseteq f(x)f(x^{-1})N.$$

But $f(N) \subseteq N$ by (i), so $f(N) \subseteq N \cap f(x)f(x^{-1})N$. Then $N = f(x)f(x^{-1})N$ which implies that $f(x^{-1})N = f(x)^{-1}N$. By (ii), $f(x^{-1}N) \subseteq f(x^{-1})N$. Hence (iv) holds.

(v) Let $x \in G$. Then by (ii), for all $k \in \mathbb{Z}$, $f(x^kN) \subseteq f(x^k)N$. It remains to show that $f(x^k)N = f(x)^kN$ for all $k \in \mathbb{Z}$. This is true for $k = 0$ and 1. Assume that $k \in \mathbb{Z}^+$ and $f(x^k)N = f(x)^kN$. Then

$$\begin{aligned} f(x^{k+1})N &= f(xx^k)N \\ &= f(x)f(x^k)N && \text{from (iii)} \\ &= f(x)(f(x^k)N) \\ &= f(x)(f(x)^kN) && \text{from the assumption} \\ &= f(x)^{k+1}N. \end{aligned}$$

This shows that $f(y^l)N = f(y)^lN$ for all $y \in G$ and $l \in \mathbb{Z}^+$. If $k \in \mathbb{Z}^+$, then

$$\begin{aligned} f(x^{-k})N &= f((x^{-1})^k)N \\ &= f(x^{-1})^kN \\ &= (f(x^{-1})N)\dots(f(x^{-1})N) && (k \text{ brackets}) \\ &= (f(x)^{-1}N)\dots(f(x)^{-1}N) && \text{from (iv)} \\ &= (f(x)^{-1})^kN \\ &= f(x)^{-k}N. \end{aligned}$$

Hence (v) is proved. ∎

**Theorem 2.2.** *For $f : \mathbb{Z} \to \mathbb{Z}$, the following statements are equivalent.*

(i) $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

(ii) $f(x + n\mathbb{Z}) \subseteq xf(1) + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

(iii) *There exists an integer $a$ such that* $f(x + n\mathbb{Z}) \subseteq xa + n\mathbb{Z}$ *for all* $x \in \mathbb{Z}$.

**Proof.** (i)⇒(ii) follows directly from Lemma 2.1(v).

(ii)⇒(iii) is evident.

(iii)⇒(i). Let $x, y \in \mathbb{Z}$. Then $f(x) \in f(x) + n\mathbb{Z}$ and $f(y) \in f(y) + n\mathbb{Z}$. Since $f(x) \in f(x + n\mathbb{Z}) \subseteq xa + n\mathbb{Z}$ and $f(y) \in f(y + n\mathbb{Z}) \subseteq ya + n\mathbb{Z}$, it follows that $f(x) + n\mathbb{Z} = xa + n\mathbb{Z}$ and $f(y) + n\mathbb{Z} = ya + n\mathbb{Z}$. Consequently,

$$\begin{aligned} f(x \circ_n y) &= f(x + y + n\mathbb{Z}) \subseteq (x + y)a + n\mathbb{Z} = xa + n\mathbb{Z} + ya + n\mathbb{Z} \\ &= f(x) + n\mathbb{Z} + f(y) + n\mathbb{Z} = f(x) + f(y) + n\mathbb{Z} = f(x) \circ_n f(y). \end{aligned}$$

Hence $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$, as desired. ∎

**Remark 2.3.** For $f : \mathbb{Z} \to \mathbb{Z}$ and $a \in \mathbb{Z}$, if $f$ and $a$ satisfies (iii) of Theorem 2.2, then $a \equiv f(1) \pmod{n}$ since $f(1) \in f(1 + n\mathbb{Z}) \subseteq a + n\mathbb{Z}$.

Recall that for any nonempty sets $X$ and $Y$,

$$|\{f \mid f : X \to Y\}| = |Y|^{|X|}$$

and in particular, if $X$ is an infinite set, then

$$|\{f \mid f : X \to X\}| = |X|^{|X|} = 2^{|X|}.$$

**Lemma 2.4.** *Let $G$ be a group and $N$ a normal subgroup of $G$. For $f \in \mathrm{Hom}(G)$, $f(N) \subseteq N$ if and only if $f \in \mathrm{Hom}(G, \circ_N)$.*

**Proof.** First, assume that $f(N) \subseteq N$. Then for all $x, y \in G$,

$$f(x \circ_N y) = f(xyN) = f(x)f(y)f(N) \subseteq f(x)f(y)N = f(x) \circ_N f(y).$$

Thus $f \in \mathrm{Hom}(G, \circ_N)$.

For the converse, assume that $f \in \mathrm{Hom}(G, \circ_N)$. Since $f \in \mathrm{Hom}(G)$, $f(e) = e$. Then

$$f(N) = f(eeN) = f(e \circ_N e) \subseteq f(e) \circ_N f(e) = f(e)f(e)N = N.$$ ∎

**Theorem 2.5.** $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, \circ_n)$.

**Proof.** Recall that

$$\mathrm{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$$

where $g_a(x) = ax$ for all $x \in \mathbb{Z}$. Since $g_a(n\mathbb{Z}) = an\mathbb{Z} \subseteq n\mathbb{Z}$ for all $a \in \mathbb{Z}$, by Lemma 2.4, $g_a \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ for all $a \in \mathbb{Z}$ and the desired result follows. ∎

From Theorem 2.5, we have $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| \geq \aleph_0$. In fact, $\mathrm{Hom}(\mathbb{Z}, \circ_n)$ is an uncountable set, as shown by the next theorem.

**Lemma 2.6.** *If $G$ is a group, then* $\mathrm{Hom}(G, \circ_G) = \{f \mid f : G \to G\}$.

**Proof.** If $f : G \to G$, then for all $x, y \in G$,

$$f(x \circ_G y) = f(xyG) = f(G) \subseteq G = f(x)f(y)G = f(x) \circ_G f(y),$$

so $f \in \mathrm{Hom}(G, \circ_G)$. Hence the result follows. ∎

**Theorem 2.7.** $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

**Proof.** By Lemma 2.6, $\mathrm{Hom}(\mathbb{Z}, \circ_1) = \{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$. Then

$$|\mathrm{Hom}(\mathbb{Z}, \circ_1)| = |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Next, assume that $n > 1$. Let $K = \{g \mid g : n\mathbb{Z} \to n\mathbb{Z}\}$. Then $|K| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Recall that for each $x \in \mathbb{Z}$, there are unique $q_x \in \mathbb{Z}$ and $r_x \in \{0, 1, \ldots, n-1\}$ such that $x = nq_x + r_x$. For each $g \in K$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(nq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for every $g \in K$, $\bar{g}_{|n\mathbb{Z}} = g$ and for $x \in \mathbb{Z}$,

$$\begin{aligned}\bar{g}(x + n\mathbb{Z}) &= \bar{g}(r_x + nq_x + n\mathbb{Z}) = \bar{g}(r_x + n\mathbb{Z}) = r_x + g(n\mathbb{Z}) \subseteq r_x + n\mathbb{Z} \\ &= r_x + nq_x + n\mathbb{Z} = x + n\mathbb{Z}\end{aligned}$$

By Theorem 2.2, we have $\bar{g} \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ for all $g \in K$. It follows that

$$\begin{aligned}2^{\aleph_0} = |K| &= |\{\bar{g} \mid g \in K\}| \\ &\leq |\mathrm{Hom}(\mathbb{Z}, \circ_n)| \\ &\leq |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}\end{aligned}$$

which implies that $|\mathrm{Hom}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

Hence the theorem is proved. ∎

## 3. Epimorphisms of the Hypergroup $(\mathbb{Z}, \circ_n)$

First, we provide the following general fact. It is used to characterize the elements of $\mathrm{Epi}(\mathbb{Z}, \circ_n)$.

**Lemma 3.1.** *Let $G$ be a group and $N$ a normal subgroup of $G$. If the index $[G:N]$ of $N$ in $G$ is finite and $f \in \mathrm{Epi}(G, \circ_N)$, then $f(xN) = f(x)N$ for all $x \in G$.*

**Proof.** Let $[G:N] = n$. Then there are $x_1, \ldots, x_n \in G$ such that $G = \bigcup_{i=1}^{n} x_i N$. Then $x_1N, \ldots, x_nN$ are mutually disjoint. By Lemma 2.1(ii), $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$. Hence

$$G = f\left(\bigcup_{i=1}^{n} x_i N\right) = \bigcup_{i=1}^{n} f(x_iN) \subseteq \bigcup_{i=1}^{n} f(x_i)N,$$

which implies that

$$G = \bigcup_{i=1}^{n} f(x_iN) = \bigcup_{i=1}^{n} f(x_i)N.$$

Since $[G:N] = n$, it follows that $f(x_1)N, \ldots, f(x_n)N$ are mutually disjoint. But $f(x_iN) \subseteq f(x_i)N$ for all $i \in \{1, \ldots, n\}$, thus we have

$$f(x_iN) = f(x_i)N \text{ for all } i \in \{1, \ldots, n\}.$$

Next, let $x \in G$. Then $xN = x_jN$ for some $j \in \{1, \ldots, n\}$. By Lemma 2.1(ii), $f(xN) \subseteq f(x)N$. Hence

$$f(x_j)N = f(x_jN) = f(xN) \subseteq f(x)N$$

which implies that $f(x)N = f(x_j)N$. Consequently,

$$f(xN) = f(x_jN) = f(x_j)N = f(x)N$$

∎

**Theorem 3.2.** *For $f : \mathbb{Z} \to \mathbb{Z}$, $f \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$ if and only if*

(i) *$f(x + n\mathbb{Z}) = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$ and*

(ii) *$f(1)$ and $n$ are relatively prime.*

**Proof.** First, assume that $f \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$. By Lemma 3.1, $f(x+n\mathbb{Z}) = f(x)+n\mathbb{Z}$ for all $x \in \mathbb{Z}$. But by Lemma 2.1(v), $f(x) + n\mathbb{Z} = xf(1) + n\mathbb{Z}$ for all $x \in \mathbb{Z}$, thus (i) holds. The fact that $f(\mathbb{Z}) = \mathbb{Z}$ and (i) yield

$$\mathbb{Z} = f\big(\bigcup_{x\in\mathbb{Z}} (x + n\mathbb{Z})\big) = \bigcup_{x\in\mathbb{Z}} (xf(1) + n\mathbb{Z}).$$

Then $1 \in yf(1) + n\mathbb{Z}$ for some $y \in \mathbb{Z}$. Thus $1 = yf(1) + tn$ for some $t \in \mathbb{Z}$ which implies that $f(1)$ and $n$ are relatively prime. Therefore (ii) holds.

For the converse, assume that (i) and (ii) hold. Then from (i) and Theorem 2.2, $f \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$. From (ii), $sf(1) + tn = 1$ for some $s, t \in \mathbb{Z}$. But since

$$\begin{aligned} \text{for every } x \in \mathbb{Z},\ x + n\mathbb{Z} &= x(sf(1) + tn) + n\mathbb{Z} \\ &= xsf(1) + n\mathbb{Z} \\ &= f(xs + n\mathbb{Z}) \qquad\qquad \text{from (i)} \\ &\subseteq f(\mathbb{Z}), \end{aligned}$$

it follows that $f(\mathbb{Z}) = \mathbb{Z}$. Hence $f \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$. ∎

To show that $|\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$, the following lemma is also needed.

**Lemma 3.3.** *If $X$ is an infinite set, then $|\{f : X \to X \mid f(X) = X\}| = 2^{|X|}$.*

**Proof.** Since $X$ is an infinite set, there are subsets $X_1$ and $X_2$ such that $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$ and $|X_1| = |X_2| = |X|$. Let $b$ and $c$ be two distinct fixed points in $X$. Then $|X_1| = |X \smallsetminus \{b\}|$. Let $\varphi : X_1 \to X \smallsetminus \{b\}$ be a bijection. For each nonempty subset $Y$ of $X_2$, define $g_Y : X \to X$ by a bracket notation as follows:

$$g_Y = \begin{pmatrix} s & Y & t \\ \varphi(s) & b & c \end{pmatrix}_{\substack{s \in X_1 \\ t \in X_2 \smallsetminus Y}}$$

Then $g_Y(X) = X$ for every nonempty subset $Y$ of $X_2$. If $Y_1$ and $Y_2$ are distinct nonempty subsets of $X_2$, then $g_{Y_1}^{-1}(b) = Y_1 \neq Y_2 = g_{Y_2}^{-1}(b)$, so $g_{Y_1} \neq g_{Y_2}$. Hence

$$\begin{aligned} 2^{|X|} = |X|^{|X|} = |\{f \mid f : X \to X\}| &\geq |\{f : X \to X \mid f(X) = X\}| \\ &\geq |\{g_Y \mid \emptyset \neq Y \subseteq X_2\}| \\ &= |\{Y \mid \emptyset \neq Y \subseteq X_2\}| \\ &= 2^{|X_2|} = 2^{|X|} \end{aligned}$$

which implies that $|\{f : X \to X \mid f(X) = X\}| = 2^{|X|}$, as desired. ∎

**Theorem 3.4.** $|\mathrm{Epi}(\mathbb{Z}, \circ_n)| = 2^{\aleph_0}$.

**Proof.** By Lemma 2.6, we have that $\mathrm{Epi}(\mathbb{Z}, \circ_1) = \{f : \mathbb{Z} \to \mathbb{Z} \mid f(\mathbb{Z}) = \mathbb{Z}\}$. Then by Lemma 3.3, $|\mathrm{Epi}(\mathbb{Z}, \circ_1)| = 2^{\aleph_0}$.

Assume that $n > 1$. Let $L = \{g : n\mathbb{Z} \to n\mathbb{Z} \mid g(n\mathbb{Z}) = n\mathbb{Z}\}$. Also, by Lemma 3.3, $|L| = 2^{\aleph_0}$. For each $x \in \mathbb{Z}$, let $q_x, r_x \in \mathbb{Z}$ be such that $r_x \in \{0, 1, \ldots, n-1\}$ and $x = nq_x + r_x$. Note that $q_x$ and $r_x$ are unique. For each $g \in L$, define $\bar{g} : \mathbb{Z} \to \mathbb{Z}$ by

$$\bar{g}(x) = r_x + g(nq_x) \text{ for all } x \in \mathbb{Z}.$$

Then for $g \in L, \bar{g}_{|n\mathbb{Z}} = g$ and we can see from the proof of Theorem 2.7 with $g(n\mathbb{Z}) = n\mathbb{Z}$ that

$$\bar{g}(x + n\mathbb{Z}) = x + n\mathbb{Z} \quad \text{for all } x \in \mathbb{Z}.$$

It follows from Theorem 2.2 that $\bar{g} \in \mathrm{Hom}(\mathbb{Z}, \circ_n)$ for all $g \in L$. We also have that

$$\bar{g}(\mathbb{Z}) = \bar{g}\big(\bigcup_{x\in\mathbb{Z}} (x+n\mathbb{Z})\big) = \bigcup_{x\in\mathbb{Z}} \bar{g}(x+n\mathbb{Z}) = \bigcup_{x\in\mathbb{Z}} (x+n\mathbb{Z}) = \mathbb{Z}.$$

Hence $\bar{g} \in \mathrm{Epi}(\mathbb{Z}, \circ_n)$ for all $g \in L$. Consequently,

$$\begin{aligned} 2^{\aleph_0} = |L| &= |\{\bar{g} \mid g \in L\}| \\ &\leq |\mathrm{Epi}(\mathbb{Z}, \circ_n)| \\ &\leq |\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}, \end{aligned}$$

so the desired result follows. ∎

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