p-FUZZY HYPERGROUPS AND p-FUZZY JOIN SPACES OBTAINED FROM p-FUZZY HYPERGRAPHS[1]

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Abstract. We construct a fuzzy hyperoperation from a p-fuzzy hypergraph and then use it to construct a p-fuzzy hypergroup and a p-fuzzy join space. Also, we study generalizations of this fuzzy hyperoperation.

Keywords: fuzzy hypergroupoid; p-fuzzy hypergroup; p-fuzzy hypergraph; p-fuzzy join space.

1. Introduction and preliminaries

The connections between graphs and hypergroups had been looked into by several researchers (see, for instance, [4], [6]). Corsini [5] and Ali [1] studied the connections between hypergraphs and hypergroups. In this paper, we construct a fuzzy hyperoperation from a p-fuzzy hypergraph and then use it to construct a p-fuzzy hypergroup and a p-fuzzy join space. Also, we study generalizations of this fuzzy hyperoperation. This paper can be seen as a fuzzy version of [5].

We recall some notations of fuzzy hyperstructure theory. A fuzzy subset of a nonempty set H is a function $M : H \to [0,1]$; The collection of all fuzzy subsets of H is denoted by $F(H)$. The p-cut of a fuzzy subset M of H is defined by

$$M_p \doteq \{x \in H \mid M(x) \geq p\}.$$

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Given a fuzzy hyperoperation $* : H \times H \to F(H)$, for all $a \in H$, $B \in F(H)$, the fuzzy subset $a * B$ of H is defined by

$$(a * B)(x) \doteq \bigvee_{B(b)>0} (a * b)(x).$$

Given $A, B \in F(H)$, we give the following definitions

$$\begin{aligned} A \subseteq B &\doteq A(x) \leq B(x), \quad \forall x \in H. \\ A = B &\doteq A(x) = B(x), \quad \forall x \in H. \\ (A \cup B)(x) &\doteq A(x) \vee B(x), \quad \forall x \in H. \\ (A \cap B)(x) &\doteq A(x) \wedge B(x), \quad \forall x \in H. \end{aligned}$$

Proposition 0.1 ([7]) $\forall A, B, C \in F(H)$, *we have the following properties*

(1) $A \cup A = A, A \cap A = A$;
(2) $A \cup B = B \cup A, A \cap B = B \cap A$;
(3) $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$;
(4) $A \cap (A \cup B) = A, A \cup (A \cap B) = A$;
(5) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C), (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
(6) $A \cup \emptyset = A, A \cap \emptyset = \emptyset, A \cup H = H, A \cap H = A$.

A fuzzy hypergroupoid $\langle H; * \rangle$ is a nonempty set H endowed with a fuzzy hyperoperation (i.e., a function $*$ from $H \times H$ to $F(H)$). A p-fuzzy quasi-hypergroup is a fuzzy hypergroupoid such that

$$(x * H)_p = H = (H * x)_p, \quad \forall x \in H.$$

A p-fuzzy hypergroup is a p-fuzzy quasi-hypergroup such that for all $x, y, z \in H$, we have

$$(x * y) * z = x * (y * z).$$

The readers can consult [2], [3], [7] to learn more about hyperstructures and fuzzy sets.

2. Fuzzy Hyperoperation $*$

Definition 2.1 H is a nonempty set, $\{A_i\}_i$ is a family of fuzzy subsets of H, if there exists a $p \in (0, 1]$ such that

$$\bigcup_i (A_i)_p = H,$$

then $\langle H; \{A_i\}_i \rangle$ is called a p-fuzzy hypergraph.

Definition 2.2 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph, set

$$E_p(x) = \bigcup_{A_i(x) \geq p} A_i.$$

The fuzzy hypergroupoid $H_\Gamma = \langle H; * \rangle$ where the fuzzy hyperoperation $*$ is defined by

$$x * y = E_p(x) \cup E_p(y), \quad \forall x, y \in H$$

is called a p-fuzzy hypergraph hypergroupoid or a p-f.h.g. hypergroupoid.

Proposition 2.3 *The p-f.h.g. hypergroupoid H_Γ has the following properties for any $x, y \in H$:*

(1) $x * y = x * x \cup y * y$;

(2) $x \in (x * x)_p$;

(3) $y \in (x * x)_p \Leftrightarrow x \in (y * y)_p$;

(4) $\{x, y\} \subseteq (x * y)_p$;

(5) $x * y = y * x$;

(6) $(x * H)_p = H$;

(7) $\langle H; \{x * x\}_{x \in H} \rangle$ *is a p-fuzzy hypergraph;*

(8) $x * x * x = \bigcup_{(x*x)(z) > 0} z * z$;

(9) $(x * x) * (x * x) = x * x * x$.

Proof.

(1) $x * y = E_p(x) \cup E_p(y) = (E_p(x) \cup E_p(x)) \cup (E_p(y) \cup E_p(y)) = x * x \cup y * y$.

(2) It is a special case of (4).

(3) Since $\bigcup_i (A_i)_p = H$, then for any $x \in H$ there exists some $A_i \in F(H)$ such that $A_i(x) \geq p$.

We only prove the implication "$\Rightarrow$". Since

$$\begin{aligned} (x * x)(y) &= (E_p(x) \cup E_p(x))(y) = (E_p(x))(y) = \left(\bigcup_{A_i(x) \geq p} A_i \right)(y) \\ &= \bigvee_{A_i(x) \geq p} A_i(y) \geq p, \end{aligned}$$

then there exists $A_i \in F(H)$ such that $A_i(x) \geq p$ and $A_i(y) \geq p$. So,

$$(y * y)(x) = \bigvee_{A_j(y) \geq p} A_j(x) \geq p.$$

Thus $x \in (y * y)_p$.

(4) $(x * y)(x) = (E_p(x) \cup E_p(y))(x) = (E_p(x))(x) \vee (E_p(y))(x) \geq (E_p(x))(x)$ $= \bigvee_{A_i(x) \geq p} A_i(x) \geq p$. So $x \in (x * y)_p$.

Similarly, we can prove $y \in (x * y)_p$.

(5) $x * y = E_p(x) \cup E_p(y) = E_p(y) \cup E_p(x) = y * x$.

(6) For any $y \in H$,

$$\begin{aligned}(x * H)(y) &= \left(\bigcup_{t \in H} x * t\right)(y) = \left(\bigcup_{t \in H} (E_p(x) \cup E_p(t)\right)(y) \\ &\geq \left(\bigcup_{t \in H} E_p(t)\right)(y) \geq (E_p(y))(y) \\ &= \left(\bigcup_{A_i(y) \geq p} A_i\right)(y) = \bigvee_{A_i(y) \geq p} A_i(y) \geq p.\end{aligned}$$

So, $H \subseteq (x * H)_p$ and thus $(x * H)_p = H$.

(7) From $x \in (x * x)_p$ we know

$$\bigcup_{x \in H} (x * x)_p = H.$$

And then $\langle H; \{x * x\}_{x \in H} \rangle$ is a p-fuzzy hypergraph.

(8) $x * x * x = \bigcup_{(x*x)(z)>0} z * x = \bigcup_{(x*x)(z)>0} (z * z) \cup (x * x) = \bigcup_{(x*x)(z)>0} z * z.$

(9) $(x * x) * (x * x) = \bigcup_{(x*x)(a)>0, (x*x)(b)>0} a * b = \bigcup_{(x*x)(a)>0, (x*x)(b)>0} (a * a \cup b * b)$ $= \bigcup_{(x*x)(a)>0} a * a = x * x * x.$ ■

Remark 2.4 From (5), (6) of the above Proposition we know that H_Γ is a commutative p-fuzzy quasi-hypergroup.

Theorem 2.5 *A p-fuzzy hypergroupoid* $\langle H; * \rangle$ *satisfying* (1), (2) *and* (3) *of Proposition* 2.3 *is a p-fuzzy hypergroup if and only if*

$$a * a * a \cup c * c = a * a * a \cup c * c * c, \quad \forall a, c \in H.$$

Proof. First, let's prove the implication "$\Leftarrow$". It is enough to verify the associativity. We have:

$(a * b) * c = (a * a \cup b * b) * c = (a * a) * c \cup (b * b) * c,$

$a * (b * c) = (b * c) * a = (b * b) * a \cup (c * c) * a, \quad \forall a, b, c \in H.$

Moreover,

$$\begin{aligned}(a * a) * c &= \bigcup_{(a*a)(u)>0} u * c = \bigcup_{(a*a)(u)>0} (u * u \cup c * c) \\ &= c * c \cup \left(\bigcup_{(a*a)(u)>0} u * u \right) = c * c \cup a * a * a.\end{aligned}$$

Also, we have

$(b * b) * c = b * b * b \cup c * c.$

Therefore,

$(a * b) * c = a * a * a \cup b * b * b \cup c * c = b * b * b \cup (a * a * a \cup c * c)$ and

$a * (b * c) = a * a \cup b * b * b \cup c * c * c = b * b * b \cup (a * a \cup c * c * c).$

By the hypothesis, we have

$a * a * a \cup c * c = a * a * a \cup c * c * c = a * a \cup c * c * c.$

And so, $(a * b) * c = a * (b * c)$.

Let's now prove the implication "$\Rightarrow$". From the associativity it follows

$(a * a) * c = a * (a * c), \quad \forall a, c \in H.$

From above we have

$(a*a)*c = a*a*a \cup c*c,\ a*(a*c) = a*a \cup a*a*a \cup c*c*c = a*a*a \cup c*c*c.$

So, $a * a * a \cup c * c = a * a * a \cup c * c * c.$ ∎

Corollary 2.6 *If a p-fuzzy hypergroupoid $\langle H; * \rangle$ satisfies* (1), (2) *and* (3) *of Proposition* 2.3 *and the condition*

$$x * x * x = x * x, \quad \forall x \in H,$$

then it is a p-fuzzy hypergroup.

Example 2.7 Let $\Gamma = \langle \{a, b\}; \{A_1, A_2\} \rangle$, where $A_1 = \dfrac{0.5}{a} + \dfrac{0.5}{b}$, $A_2 = \dfrac{0.5}{a} + \dfrac{0.5}{b}$. Since $\bigcup_{i=1}^{2} (A_i)_{0.5} = (A_1)_{0.5} \cup (A_2)_{0.5} = \{a, b\} \cup \{a, b\} = \{a, b\}$, then Γ is a 0.5-fuzzy hypergraph. Moreover,

$a * a = b * b = a * b = b * a = a * a * a = b * b * b = \dfrac{0.5}{a} + \dfrac{0.5}{b}.$

So, from above Corollary we know that $\langle \{a, b\}; * \rangle$ is a 0.5-fuzzy hypergroup.

Example 2.8 Let $\Gamma = \langle \{a, b\}; \{A_1, A_2\}\rangle$, where $A_1 = \dfrac{0.5}{a} + \dfrac{0.8}{b}$, $A_2 = \dfrac{0.7}{a} + \dfrac{0.2}{b}$. Since $\bigcup_{i=1}^{2}(A_i)_{0.5} = (A_1)_{0.5} \cup (A_2)_{0.5} = \{a, b\} \cup \{a\} = \{a, b\}$, then Γ is a 0.5-fuzzy hypergraph. Moreover,

$$a * a = E_{0.5}(a) = A_1 \cup A_2 = \frac{0.7}{a} + \frac{0.8}{b}.$$

$$b * b = E_{0.5}(b) = A_1 = \frac{0.5}{a} + \frac{0.8}{b}.$$

$$a * a * a = \left(\frac{0.7}{a} + \frac{0.8}{b}\right) * a = a * a \cup a * b = \frac{0.7}{a} + \frac{0.8}{b}.$$

$$b * b * b = \left(\frac{0.5}{a} + \frac{0.8}{b}\right) * b = a * b \cup b * b = \frac{0.7}{a} + \frac{0.8}{b}.$$

We have $b * b * b \neq b * b$.

But $x * x * x \cup y * y = x * x * x \cup y * y * y,\ \forall x, y \in \{a, b\}$.

So, from Theorem 2.5, we know that $\langle \{a, b\}; *\rangle$ is a 0.5-fuzzy hypergroup.

Definition 2.9 An associative p-f.h.g. quasi-hypergroup is called a p-f.h.g. hypergroup.

Definition 2.10 Let $\langle H; *\rangle$ be a commutative p-fuzzy hypergroup, $\langle H; *, /\rangle$ is called a p-fuzzy join space if and only if

$$(x/y \cap z/w)_p \neq \emptyset \Rightarrow (x * w \cap y * z)_p \neq \emptyset$$

where $(x/y)(t) = (t * y)(x)$.

Theorem 2.11 *Let $\langle H; *\rangle$ be a p-fuzzy hypergroup satisfying* (1), (2) *and* (3) *of Proposition* 2.3. *Then $\langle H; *, /\rangle$ is a p-fuzzy join space.*

Proof. We prove the following implication is valid:

$$(x/y \cap z/w)_p \neq \emptyset \Rightarrow (x * w \cap y * z)_p \neq \emptyset \text{ where } (x/y)(t) = (t * y)(x).$$

We have

$$u \in (x/y \cap z/w)_p \Leftrightarrow [x \in (u * y)_p \text{ and } z \in (u * w)_p].$$

Moreover,

$$x \in (u * y)_p \Leftrightarrow x \in (u * u \cup y * y)_p = (u * u)_p \cup (y * y)_p \text{ and}$$

$$z \in (u * w)_p = z \in (u * u \cup w * w)_p = (u * u)_p \cup (w * w)_p.$$

Four cases are possible:

(1) if $x \in (u * u)_p, z \in (u * u)_p$, then $u \in (x * x)_p \cap (z * z)_p = (x * x \cap z * z)_p$ and therefore $u \in (x * w \cap y * z)_p$.

(2) if $x \in (u * u)_p, z \in (w * w)_p$, then $w \in (z * z)_p$ and therefore $w \in (x * w \cap y * z)_p$.

(3) if $x \in (y * y)_p, z \in (u * u)_p$, then $y \in (x * x)_p$ and therefore $y \in (x * w \cap y * z)_p$.

(4) if $x \in (y * y)_p, z \in (w * w)_p$, then $w \in (z * z)_p$ and therefore $w \in (x * w \cap y * z)_p$. ■

3. Generalizations of $*$

We can generalize the fuzzy hyperoperation $*$ in following ways.

Definition 3.1 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph, for all $q \in (0, p]$, set

$$E_q(x) = \bigcup_{A_i(x) \geq q} A_i$$

and the fuzzy hyperoperation $*_q$ is defined by

$$x *_q y = E_q(x) \cup E_q(y), \quad \forall x, y \in H.$$

Proposition 3.2 *The fuzzy hyperoperation $*_q$ has the following properties for any $x, y \in H$:*

(1) $x *_q y = x *_q x \cup y *_q y$;

(2) $x \in (x *_q x)_p$;

(3) $y \in (x *_q x)_q \Leftrightarrow x \in (y *_q y)_q$;

(4) $\{x, y\} \subseteq (x *_q y)_p$;

(5) $x *_q y = y *_q x$;

(6) $(x *_q H)_p = H$;

(7) $\langle H; \{x *_q x\}_{x \in H} \rangle$ *is a p-fuzzy hypergraph;*

(8) $x *_q x *_q x = \bigcup_{(x *_q x)(z) > 0} z *_q z$;

(9) $(x *_q x) *_q (x *_q x) = x *_q x *_q x$.

Proof. A straightforward verification. ■

Definition 3.3 Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be a p-fuzzy hypergraph. For all $q \in (0, p]$, set

$$E_q(x) = \bigcup_{A_i(x) \geq q} A_i.$$

For all $s, t \in (0, p]$, the fuzzy hyperoperation $*_s^t$ is defined by

$$x *_s^t y = E_s(x) \cup E_t(y), \quad \forall x, y \in H.$$

Proposition 3.4 *The fuzzy hyperoperation $*_s^t$ has the following properties for any $x, y \in H$:*

(1) $x *_s^t y = x *_s^s x \cup y *_t^t y$;

(2) $x \in (x *_s^t x)_p$;

(3) $y \in (x *_s^t x)_{s \vee t} \Leftrightarrow x \in (y *_s^t y)_{s \vee t}$;

(4) $\{x, y\} \subseteq (x *_s^t y)_p$;

(5) $x *_s^t y = y *_t^s x$;

(6) $(x *_s^t H)_p = H$;

(7) $\langle H; \{x *_s^t x\}_{x \in H}\rangle$ *is a p-fuzzy hypergraph;*

Proof. A straightforward verification. ∎

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