# GENERALIZATION OF GOLDBACH'S CONJECTURE AND SOME SPECIAL CASES

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**Abstract.** Concerned with Goldbach's conjecture, we accomplished a generalization that we called *generalized Goldbach's conjecture* and proved their equivalency. However, the generalized Goldbach's conjecture reveals a new direction for a potential generalized proof. In this paper we prove both claims for certain cases.

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## 1. Introduction

The Goldbach's conjecture [4],

> *every even positive integer number* (*i.e., every positive integer multiple of* 2) *besides* 2*, is analyzed* (*not necessarily uniquely*) *as the sum of two positive prime numbers*[1],

gives rise to the question whether a similar conjecture can be stated for the positive multiples of every positive integer number. In particular whether

> *every positive multiple of every positive integer $a$, except itself, is analyzed* (*not necessarily uniquely*) *as the sum of $a$ prime numbers.*

By considering simple examples, it is confirmed that the statement holds. Since it is unproved, it remains a simple conjecture and because it generalizes Goldbach's conjecture we characterized it as *generalized Goldbach's conjecture*. However, considering their elaborations we have concluded that Goldbach's conjecture implies the generalized Goldbach's conjecture and vice versa. It is worth

[1]The statement concerns the binary conjecture also known as the *strong Goldbach's conjecture*, in contrast to Goldbach's original ternary conjecture which states that every odd integer number greater or equal to seven is the sum of three primes.

noting that Goldbach's conjecture has received a lot of attention; see, for example, [1], [2], [3], [5], [6], [7].

For proving the equivalency of both conjectures we will proceed inductively. From the beginning, let us assume the division

$$am/(a-1) \tag{1}$$

of the positive multiple $am$ of $a$ by $a-1$, that is the relationship

$$am = (a-1)n + r$$

where $n$ and $r$ are the quotient and the remainder, respectively. Thus we have $0 \leq r < a-1$. Next we assume that $a-1$ is analyzed as the sum of $a-1$ prime numbers and we will prove the correspondence for $a$.

## 2. Base cases $a = 1, 2, 3, 4, 5$ for the induction

The case for $a = 1$ is obviously excluded, unless $m$ prime, for $a = 2$ we have Goldbach's conjecture that we accept as a proved statement or as an axiom. Thus we have $a \geq 3$ and for the induction we will consider the following cases ($a = 3, 4, 5$).

### 2.1. Case $a = 3$

From the division $3m/2$ we have the cases $3m = 2n + 1$ and $3m = 2n$.

(3.$i$) $3m = 2n + 1 = p_1 + q_1 + 1$, where $p_1 + q_1$ by the analysis of Goldbach of $2n$ as the sum of two prime numbers (for $n > 1$ and, thus, for $m > 1$) and because we exclude $p_1 = q_1 = 2$ (for otherwise we would have $3m = 5$, not a multiple of 3) one of $p_1, q_1$ will be odd number. Let such a number be $q_1$. Then $q_1 + 1$ is even and by Goldbach $q_1 + 1 = p_2 + p_3$ sum of two primes. Hence we conclude that $3m = p_1 + p_2 + p_3$ sum of three prime numbers.

**Example 2.1.**

$$3 \cdot 7 = 2 \cdot 10 + 1 = 21 = 2 + 2 + 17 = 3 + 5 + 13 = 5 + 5 + 11 = 7 + 7 + 7.$$

(3.$ii$) $3m = 2n = 2(n-1) + 2 = p_1 + p_2 + 2$, where $p_1 + p_2$ one analysis of $2(n-1)$ as the sum of two prime numbers, since $n - 1 > 1$. This means for $n > 2$ we have $m > 1$.
In such a case $3m$ is obviously an even multiple of 3. (Moreover we have $3m = 2n = 6m'$ because 2 as a divisor of $3m$ and prime number with respect to 3 divides $m$ and thus $m = 2m'$.)

**Example 2.2.**

$$3 \cdot 6 = 3(2 \cdot 3) = 2 \cdot 9 = 18 = 2 + 3 + 13 = 2 + 5 + 11.$$

**Remark 2.1.**

a) In every analysis that concerns (as it is non-unique) every even multiple of 3 into a sum of three prime numbers, one of them is 2.

b) The above analysis is unique only if $p_1 = p_2 = p_3 = 2$, in which case we have $2 + 2 + 2 = 3 \cdot 2$. Thus it holds for $m = 2$ which is true for $a \in \mathbb{N}$: $a \cdot 2 = 2 + 2 + \cdots + 2$, sum of $a$ numbers.

c) Moreover it follows that for every $a \in \mathbb{N}$ there is no additive analysis for the multiple of $am$ for $m = 1$, i.e., for $a$ itself, as the sum of $a$ prime numbers. Thus we will assume in general that $m \geq 2$.

### 2.2. Case $a = 4$

We have (from the division $4m/3$) the case $4m = 3n + 1$, $4m = 3n + 2$, and $4m = 3n$.

(4.$i$) $4m = 3n + 1 = p_1 + p_2 + p_3' + 1$, where $p_1, p_2, p_3'$ are prime numbers of an additive analysis of $3n$ and, as before, the case for which $p_1 = p_2 = p_3' = 2$, is excluded. At least one of them (more precisely two) will be odd number. Let such a number be $p_3'$. Then $p_3' + 1$ is even and thus $p_3' + 1 = p_3 + p_4$ is sum of two prime numbers. Hence $4m = p_1 + p_2 + p_3 + p_4$ is an additive analysis of $4m$ into four additive prime numbers.

**Example 2.3.**

$$\begin{aligned} 4 \cdot 4 &= 16 = 3 \cdot 5 + 1 = (2 + 2 + 11) + 1 = 2 + 2 + 5 + 7 \\ &= 3 + 3 + 3 + 7 = 3 + 3 + 5 + 5. \end{aligned}$$

$$\begin{aligned} 4 \cdot 7 &= 28 = 3 \cdot 9 + 1 = 2 + 2 + 5 + 19 = 3 + 3 + 3 + 19 \\ &= 3 + 3 + 5 + 17 = 5 + 5 + 5 + 13 = 7 + 7 + 7 + 7. \end{aligned}$$

(4.$ii$) $4m = 3n + 2 = p_1 + p_2 + p_3 + 2$ is the sum of four primes one of which is 2. We have then that (excluding the case $p_1 = p_2 = p_3 = 2$ that holds for $4m = 4 \cdot 2$, i.e., $m = 2$) since the sum $p_1 + p_2 + p_3$ is even with three additives, one of them must be 2. Let such a number be $p_3 = 2$. Finally we have $4m = p_1 + p_2 + 2 + 2$ being the sum of four prime numbers, not only in this form (where two additives being 2), as we have the following examples.

**Example 2.4.**

$$\begin{aligned} 4 \cdot 5 &= 20 = 3 \cdot 6 + 2 = 2 + 2 + 3 + 13 \\ &= 2 + 2 + 5 + 11 = 3 + 3 + 7 + 7. \end{aligned}$$

$$\begin{aligned} 4 \cdot 8 &= 32 = 3 \cdot 10 + 2 = 2 + 2 + 5 + 23 = 2 + 2 + 11 + 17 \\ &= 3 + 3 + 7 + 19 = 3 + 3 + 3 + 23 = 5 + 5 + 5 + 17 = 7 + 7 + 7 + 11. \end{aligned}$$

(4.$iii$) $4m = 3n = 3(n-1)+3 = p_1+p_2+p_3+3$ is the additive analysis of $4m$ as sum of four primes one of which is 3. Moreover, we have

$$4m = 3n = 12m' = (12m'-3)+3 = 3(4m'-1)+3 = p_1+p_2+p_3+3.$$

However, there are analysis of $4m$ without necessarily one of the four additives being 3. Indeed, we have

$$\begin{aligned} 4m = 3n &= 12m' = (12m'-2)+2 = 2(6m'-1)+2 = p_1'+p_2'+2 \\ &= (p_1'+1)+(p_2'+1) = p_1+p_2+p_3+p_4, \end{aligned}$$

excluding the case for which $p_1' = p_2' = 2$. The numbers $p_1', p_2'$ are odd primes and thus the sums $p_1'+1, p_2'+1$ are even numbers.

**Example 2.5.**

$$\begin{aligned} 4\cdot 6 &= 3\cdot 8 = 12\cdot 2 = (12\cdot 2-2)+2 = 2(6\cdot 2-1)+2 = (19+3)+2 \\ &= (17+5)+2 = (11+11)+2 = (19+1)+(3+1) \\ &= (17+1)+(5+1) = (11+1)+(11+1) = 20+4 = 18+6 \\ &= 12+12 = (17+3)+(2+2) = (13+7)+(2+2) \\ &= (13+5)+(3+3) = (11+7)+(3+3) = (5+7)+(5+7) \\ &= 2+2+3+17 = 2+2+7+13 = 3+3+5+13 \\ &= 3+3+7+11 = 5+5+7+7. \end{aligned}$$

Furthermore, we have $3+3+5+13 = 3+5+5+11$ and $2+2+3+17 = 3+7+7+7$. Finally, we obtain

$$\begin{aligned} 4\cdot 6 &= 2+2+3+17 = 2+2+7+13 \\ &= 3+3+5+13 = 3+3+7+11 \\ &= 5+5+7+7 = 3+5+5+11 = 3+7+7+7. \end{aligned}$$

### 2.3. Case $a = 5$

We distinguish the following cases according to the remainders of the division $5m/4$:

$$5m = 4n+1,\ 5m = 4n+2,\ 5m = 4n+3,\ 5m = 4n = 20m'.$$

We assume an additive analysis of $4n$ as the sum of four primes. Excluding the case $4n = 2+2+2+2$, we have

(5.$i$) $5m = 4n+1 = (p_1+p_2+p_3+p_4')+1 = p_1+p_2+p_3+(p_4'+1) = p_1+p_2+p_3+p_4+p_5$, $p_4'+1 = p_4+p_5$ sum of two primes as an even number ($p_4'$ is an odd number).

(5.*ii*) $5m = 4n + 2 = (p_1 + p_2 + p_3 + p_4) + 2$ is the sum of five prime numbers one of which being 2.

(5.*iii*) $5m = 4n + 3 = (p_1 + p_2 + p_3 + p_4) + 3$ is the sum of five prime numbers one of which being 3 (or and another analysis as in the case (5.*i*)).

(5.*iv*) $5m = 4n = 20m' = (20m' - 4) + 4 = p_1' + p_2' + 4 = (p_1' + 1) + (p_2' + 1) + 2 = p_1 + p_2 + p_3 + p_4 + 2$ as in the case (5.*ii*) (where $20m' - 4 = p_1' + p_2'$, sum of two primes being an even number).

**Remark 2.2.**

By considering the case $a = 5$ (and the previous) it is obvious that instead of examining separately each of the cases 0, 1, 2, 3, 4 of the remainder from the division $5m/4$, it is enough to consider the cases of even $2k$ and odd $2k + 1$ remainders. That is, for the cases

$$5m = 4n + 2k, \text{ and } 5m = 4n + 2k + 1.$$

## 3. Goldbach's conjecture implies generalized Goldbach's conjecture

Next, we proceed inductively in order to generalize the additive analysis of every positive multiple of $am$ as the sum of $a$ prime numbers for $a \in \mathbb{N} \setminus \{2\}$ according to the previous facts and assuming the validity of Goldbach's conjecture. We assume the division (1) and distinguish its remainders for the multipliers $m \in \mathbb{N}$ with respect to even $2k$ and prime $2k + 1$ numbers. We have two cases, $a$ being an even number and $a$ being an odd number.

**I.** $a$ **even.** We have the cases

$$\begin{aligned} am &= (a-1)n + 2k \quad \text{and} \\ am &= (a-1)n + 2k + 1. \end{aligned}$$

i. Let $am = (a-1)n + 2k$. It is obvious that $am$ and $2k$ are even numbers and thus $(a-1)n$ is even. According to the induction hypothesis we have $(a-1)n = p_1 + \cdots + p_{a-1}$ is a sum of additives prime numbers where their sum $(a-1)n$ is even and thus

$$am = (p_1 + \cdots + p_{a-1}) + 2k.$$

Because the number of the additives is odd and their sum is even, it follows that at least one of $p_1, \ldots, p_{a-1}$, is even, and more precisely being 2. Let such a number be $p_{a-1}$. Then,

$$am = (p_1 + \cdots + p_{a-2}) + 2 + 2k.$$

The sum $2+2k$ is even and according to Goldbach's conjecture, $2+2k = p+q$ is the sum of two prime numbers. Hence

$$\begin{aligned} am &= (p_1 + \cdots + p_{a-2}) + p + q \\ &= p_1 + \cdots + p_{a-2} + p_{a-1} + p_a, \end{aligned}$$

sum of $a$ additive primes for each of its multiplier.

**Remark 3.1.**

a) We observe that in the two assumed additive analysis of $(a-1)n$ and $am$ as sum of primes, $a-2$ additives are common. These two additive analysis where the one of $am$ follows from that of $(a-1)n$ with the above procedure, are called *corresponding*.

b) In the above case of the additive analysis

$$(a-1)n = p_1 + \cdots + p_{a-1}$$

we can have not only one but an odd number of the additives being equal to 2, depending of course by the multiplier $m$. Because $a-1$ is odd, in the special case where each of the additives is 2, we have $(a-1)n = (a-1)2$, and thus $n = 2$. By $am = 2(a-1)+2k$ assumed as division $am/2$, we have $2k = 2$, and thus $am = 2(a-1)+2 = 2a$ and $m = 2$.

c) In the case for which we have $am = (a-1)n + 2k$ and $k = 0$ then (likewise in the special cases for $a = 3, 4, 5$) we have

$$am = (a-1)n = a(a-1)n'$$

(because $a-1$ is prime with respect to $a$ and divides $m$ so that $m = (a-1)n'$). Hence,

$$\begin{aligned} am &= (a-2)an' + an' \\ &= p_1 + \cdots + p_{a-2} + an'. \end{aligned}$$

Since $an'$ is even, we have $an' = p + q$, and thus again

$$am = p_1 + \cdots + p_{a-2} + p + q,$$

is sum of $a$ prime additives.

ii. Let $am = (a-1)n + 2k + 1$. According to the induction hypothesis we have $(a-1)n = p_1 + \cdots + p_{a-1}$ and thus

$$am = p_1 + \cdots + p_{a-1} + 2k + 1,$$

where we have an odd number of additives (since $am$ is even for every $m$). Then at least one of the additives is odd (or an odd number of them). Let such a number be $p_{a-1}$. Then the sum $p_{a-1}+2k+1$ is even and thus $p_{a-1}+2k+1=p+q$. Hence,

$$\begin{aligned} am &= p_1+\cdots+p_{a-2}+p+q \\ &= p_1+\cdots+p_{a-2}+p_{a-1}+p_a, \end{aligned}$$

is the sum of $a$ prime additives.

**II.** $a$ **odd** We have again the cases

$$\begin{aligned} am &- (a-1)n+2k \quad \text{and} \\ am &= (a-1)n+2k+1. \end{aligned}$$

It is obvious that the first case occurs only for the even multipliers of $a$ and the second one only for the odds (because $a-1$ is an even number). Moreover we have in both cases by the induction hypothesis the additive analysis

$$(a-1)n = p_1+\cdots+p_{a-1}$$

with an even number of prime additives.

i. Let $am=(a-1)n+2k$ with $m=2m'$ (even). We exclude the case for which $p_1=\cdots=p_{a-1}=2$, so that

$$am=(a-1)2+2k$$

and as before we have $k=1$, that is, we exclude the case $am=(a-1)2+2$, so that $m=2$. For every other even multiple of $a$ we have that at least two of the additives $p_1,\ldots,p_{a-1}$ are odd numbers (or an even number of them less than $a-1$). Let them be $p_{a-2},p_{a-1}$. Then we have,

$$\begin{aligned} am &= (p_1+\cdots+p_{a-3})+p_{a-2}+p_{a-1}+(2k-2)+2 \\ &= p_1+\cdots+p_{a-3}+2+(p_{a-2}+p_{a-1}+2k-2) \\ &= p_1+\cdots+p_{a-3}+2+p+q, \end{aligned}$$

where $p+q$ is an analysis of the even number $p_{a-2}+p_{a-1}+2k-2$ as sum of two primes according to Goldbach's conjecture. Hence $am$ is written as the sum of $a$ prime numbers.

**Remark 3.2.**

a) As in the corresponding case for even $a$, in the additive analysis of $(a-1)n$ an even number of additives can be odd numbers. For instance,

$$\begin{aligned} am = 7\cdot 10 = 70 &= 6\cdot 11+2+2 \\ &= 3+5+13+13+17+17+2 \\ &= 2+2+13+13+19+19+2 \\ &= 2+2+2+23+13+13+13+2. \end{aligned}$$

b) In every analysis of an even multiplier of an odd number as the sum of primes, at least one of the additives is 2.

c) If $am = (a-1)n$ ($a$ odd, $m$ even, $k = 0$), then we have

$$am = (a-3)n + 2n = (a-3)n + 2(n-1) + 2.$$

Hence

$$am = p_1 + \cdots + p_{a-3} + p + q + 2,$$

is the sum of $a$ prime numbers ($2(n-1) = p + q$, since $n > 1$).

ii. Let $am = (a-1)n + 2k + 1$ with $m$ being an odd number. Then, by the induction hypothesis,

$$am = p_1 + \cdots + p_{a-1} + 2k + 1$$

where the sum $p_1 + \cdots + p_{a-1}$ is an even number. If $p_1 = \cdots = p_{a-1} = 2$ then

$$\begin{aligned} am &= (a-3)2 + 2 + 2 + 2k + 1 \\ &= (a-3)2 + 3 + (2 + 2k) \\ &= (a-3)2 + 3 + p + q \\ &= (2 + \cdots + 2) + 3 + p + q. \end{aligned}$$

Otherwise, there exists an even number of additives different than 2. If $p_{a-1}$ is one of them, we have

$$\begin{aligned} am &= (p_1 + \cdots + p_{a-2}) + p_{a-1} + 2k + 1 \\ &= p_1 + \cdots + p_{a-2} + p + q, \end{aligned}$$

which is the sum of $a$ prime numbers, because the number $p_{a-1} + 2k + 1$ is even.

From the previous facts we obtain the following fundamental theorem (that we already mentioned in the Introduction).

**Theorem 3.1.** *The axiomatic acceptance of Goldbach's conjecture implies the validity of the generalized conjecture.*

From the proof of the theorem and as a starting point the division $am/(a-1)$, i.e., the relationship

$$am = (a-1)n + r,$$

and according to Goldbach's conjecture, we devise several properties of the generalized conjecture, some of which are mentioned in their proper positions but others have not been mentioned. For that reason we conclude all of them in the following proposition.

**Proposition 3.1.** *In every additive analysis of* $(a-1)n$ *as a sum of* $a-1$ *additive primes corresponds an analysis of* $am$ *as the sum of* $a$ *prime numbers from which the* $a-2$ *are common in both those analysis, except the case* $a$ *being odd and* $r=2k$ *for which the common additives are* $a-3$*. In such a case at least two of the even number of prime additives of the analysis of* $a-1$ *are different than* 2 (*possible all of them*)*. In such an additive analysis of* $a-1$ *if* $a$ *is even and* $r=2k$ *then one of the additives is* 2 *.*

**Remark 3.3.** Therefore by Goldbach's conjecture we conclude not only the generalized Goldbach's conjecture but we obtain also the properties of Proposition 3.1.

**Corollary 3.1.** *By the acceptance of Goldbach's conjecture we have that every integer* $a \in \mathbb{N}$ *admits a number of additives analysis into the sum of primes which is equal to the number of its divisibles with the number of additives in each one* (*not necessarily unique*) *as the corresponding divisor.*

**Example 3.1.** Let $a = 12$. Then the divisors are $(1), 2, 3, 4, 6, (12)$.

$$\begin{array}{lcl} \text{For } a = 2 & \text{we have} & 12 = 5+7. \\ \text{For } a = 3 & \text{we have} & 12 = 2+3+7 = 2+5+5. \\ \text{For } a = 4 & \text{we have} & 12 = 2+2+3+5 = 3+3+3+3. \\ \text{For } a = 6 & \text{we have} & 12 = 2+2+2+2+2+2. \end{array}$$

## 4. Generalized Goldbach's conjecture implies Goldbach's conjecture

For the converse, now we examine whether the axiomatic acceptance of the generalized Goldbach's conjecture and Proposition 3.1 for numbers $a \in \mathbb{N} \smallsetminus \{2\}$ imply Goldbach's conjecture.

In order to show the validity of the generalized Goldbach's conjecture we focused in each of the cases of the relationship

$$am = (a-1)n + r, \; 0 \leq r < a-1,$$

for $a$ even with $r = 2k$ and $r = 2k+1$, and for $a$ odd with $r = 2k$ and $r = 2k+1$. We also proceed in a similar fashion.

Let $2s$ be a positive multiplier of 2 with $s \neq 1$ and let an even integer $a$ such that $a > 2s+1$. We consider a proper multiple $am$ of $a$ so that the remainder of the division $am/(a-1)$ is $2(s-1)$ (for instance, this is determined with an indefinite analysis of the equation $ax - (a-1)y = 2(s-1)$ ). If $n$ is the quotient, we have

$$am = (a-1)n + 2(s-1).$$

Because the product $(a-1)n$ is an even number ($a$ is even and $2(s-1)$ is even), the additive analysis of $(a-1)n = p_1 + \cdots + p_{a-1}$ has an odd number of prime additives one of which is 2. Let such a number be $p_{a-1}$. Thus we have

$$am = p_1 + \cdots + p_{a-2} + 2 + 2(s-1) = p_1 + \cdots + p_{a-2} + 2s.$$

Since $am$ is analyzed into a sum of $a$ prime additives from which $a-2$ are according to Proposition 3.1 additives of the additive analysis of $(a-1)n$ into $a-1$ primes, the rest two primes $p$ and $q$ of the analysis of $am$ are expressed by the remainder $2s$. This means that

$$am = p_1 + \cdots + p_{a-2} + p + q$$

and thus

$$2s = p + q,$$

which is exactly Goldbach's conjecture. Therefore we prove the following important theorem.

**Theorem 4.1.** *If for the positive multiples of any two numbers* $a,\ a-1 \in \mathbb{N} \smallsetminus \{2\}$, *the generalized Goldbach's conjecture holds, then so does the Goldbach's conjecture.*

## 5. Proofs of the conjectures for some cases

The generalized Goldbach's conjecture opens a new direction for proving Goldbach's conjecture. Indeed it remains to prove that the positive multiples of any two consecutive integers $a$ and $a-1$ can be analyzed as sums of $a$ and $a-1$, respectively, additive primes. It is natural to expect that the effort begins with the numbers of the smallest pair $(4,3)$ of consecutives $(a, a-1)$. The smaller pair $(3,2)$ is excluded from the effort, since analyzing its numbers implies the proof for Goldbach's conjecture. Thus we have,

### 5.1 Case $a = 3$

In order to apply induction we let at the beginning multiples of 3 with additive analysis $p_1 + p_2 + p_3$ in which two of the three prime numbers to be the smallest pairs of integers $(2,2), (2,3)$.

$$\begin{aligned}
3 \cdot 3 &= 3(2+1) = 9 = 2+2+5 \\
3 \cdot 4 &= 3(2 \cdot 2) = 12 = 2+3+7 \\
3 \cdot 5 &= 3(2 \cdot 2 + 1) = 15 = 2+2+11 \\
3 \cdot 6 &= 3(2 \cdot 3) = 18 = 2+3+13
\end{aligned}$$

And generalizing by induction:

$$\begin{aligned}
3(2m+1) &= 2+2+p \ (p \text{ prime, thus } m \geq 1) \\
3(2m) &= 2+3+p \ (p \text{ prime, thus } m \geq 2)
\end{aligned}$$

Furthermore, we obtain

a) $\qquad 3(2m+1) = 2+2+p \Rightarrow p = 6m+3-4 = 6m-1$, true.

Because, as it is known from Number Theory, *every positive prime number different than* 2 *and* 3 *is of the form* mul6 + 1 *or* mul6 − 1. The converse however does not hold[2] (for instance, $4 \cdot 6 + 1$, $6 \cdot 6 - 1$ are not prime numbers).

For the converse now, for each positive prime $p$ of the form $6m-1$ corresponds a (positive) odd multiple of 3 with an additive analysis $2 + 2 + p$. Indeed,

$$p = 6m - 1 \Rightarrow 6m = p + 1 \Rightarrow 6m + 3 = p + 4$$

and thus

$$3(2m + 1) = 2 + 2 + p.$$

Moreover every odd multiple of 3 does not admit the additive analysis $2 + 2 + p$ where $p$ is prime. In order to admit such an analysis, we need the difference

$$3(2m + 1) - (2 + 2) = 6m + 3 - 4 = 6m - 1$$

to be prime number. For instance, for $3(2m+1) = 3(2\cdot 2+1)$ we have $6\cdot 2-1 = 11$ which is a prime number. And thus

$$3(2 \cdot 2 + 1) = 15 = 2 + 2 + 11.$$

Whereas for $3(2m + 1) = 3(2 \cdot 6 + 1)$ we have $6 \cdot 6 - 1 = 35$ which is a non-prime number and hence the multiple $3(2 \cdot 6 + 1)$ of 3 does not admit such an analysis $2 + 2 + p$ where $p$ is prime. Therefore we have the following theorem.

**Theorem 5.1.** *The class of the odd multiples* $3(2m+1)$ *of* 3 *for which the number* $6m-1 = p$ *is prime, verifies the generalized Goldbach's conjecture with an additive analysis* $2 + 2 + p$ *for each such a multiple.*

Related to the odd multiples of 3, we observe that every such number is odd and for which only one class of numbers according to the theorem admits an additive analysis $2 + 3 + p$ for $p$ prime. Thus we conclude for the odd numbers the following theorem.

**Theorem 5.2.** *The class of odd numbers of the form* $3(2m + 1)$ *for which the number* $6m - 1 = p$ *is prime, verifies the Goldbach's conjecture* (*for the odd numbers*) *with an additive analysis* $2 + 3 + p$ *for each such number.*

Indeed

$$3(2m + 1) = 6m + 3 = (6m - 1) + 4 = p + 2 + 3.$$

For instance, for $3(2m + 1) = 3(2 \cdot 1 + 1)$ we have $6 \cdot 1 - 1 = 5 = p$ prime number and thus $3(2 \cdot 1 + 1) = 9 = 2 + 2 + 5$. Whereas for $3(2m + 1) = 3(2 \cdot 6 + 1)$ we have $6 \cdot 6 - 1 = 35$ non-prime and thus the number $3(2 \cdot 6 + 1) = 39$ does not

[2] Because one of the numbers $p - 1, p, p + 1$ is divisible by 3 we have either $p + 1 =$mul3 or $p-1 =$mul3, since $p$ is prime and different than 3. Moreover, each of the $p-1$, $p+1$ is even and if it is also multiple of 3 then it is also multiple of 6. Thus we have $p =$mul6+1 or $p =$mul6−1.

admit an additive analysis $2+2+p$, for $p$ prime. It admits however according to Goldbach's conjecture, other kind of analysis; for instance,

$$\begin{aligned} 39 &= 3+5+31 = 3+7+29 = 3+13+23 \\ &= 3+17+19 = 5+5+29 = \\ &= 5+11+23 = 5+17+17 = 7+13+19 \\ &= 17+11+11 = 13+13+13. \end{aligned}$$

b) $\quad 3(2m) = 2+3+p \Rightarrow p = 6m-5 = (6m-6)+1 = 6(m-1)+1$ true,

according to the above facts. For the converse, for every positive prime number of the form $p = 6m+1$ there is an even multiple of 3 with an additive analysis $2+3+p$. Indeed,

$$p = 6m+1 \Rightarrow 6m = p-1 \Rightarrow 6m+6 = p+5$$

and thus

$$3\left(2(m+1)\right) = 2+3+p.$$

In a similar fashion with the case above, every even multiple of 3 does not admit an analysis of the form $2+3+p$. In order to admit such an analysis, we need the difference

$$\begin{aligned} &3(2m)-(2+3) \qquad \text{or} \\ &6m-5 = (6m-6)+1 = 6(m-1)+1 \end{aligned}$$

to be a prime number. For instance, for $3(2m) = 3(2\cdot 3)$ we have $6\cdot 2+1 = 13$ prime number and thus

$$3(2\cdot 3) = 18 = 2+3+13.$$

Whereas for $3(2m) = 3(2\cdot 9)$ we have $6\cdot 8+1 = 49$ non-prime number and thus the multiple $3(2\cdot 9)$ of 3 does not admit the analysis $2+3+p$ where $p$ is a prime number.

Therefore in correspondence with Theorem 5.1, we have the following:

**Theorem 5.3.** *The class of the even multiples* $3(2m)$ *of* 3 *for which the number* $6(m-1)+1 = p$ *is prime, verifies the generalized Goldbach's conjecture with an additive analysis* $2+3+p$ *for each such a multiple.*

Suppose now the even multiple $3(2m)$ of 3 admits an additive analysis $2+3+p$ where $p$ is prime, i.e., the relationship $3(2m) = 2+3+p$ with $p = 6(m-1)+1$ form the previous theorem. By such an equivalence we have that $2(3m) = 5+p$ which means that we refer to Goldbach's conjecture with the following important theorem:

**Theorem 5.4.** *The class of the even number of the form* $2(3m)$ *for which the number* $6(m-1)+1 = p$ *is prime, verifies Goldbach's conjecture with an additive analysis* $5+p$ *for each such a number.*

Indeed
$$2(3m) = 6m = (6(m-1)+1)+5 = 5+p.$$
For instance, for $2(3m) = 2(3\cdot 3)$ we have $6(3-1)+1 = 13 = p$ prime number and thus $2(3\cdot 3) = 18 = 5+13$. Whereas for $2(3m) = 2(3\cdot 5)$ we have $6(5-1)+1 = 25$ non-prime number and the number $2(3\cdot 5) = 30$ does not admit the analysis $5+p$ but according to Goldbach's conjecture it admits several others (in the considered case we have $30 = 7+23 = 11+19$).

Especially for the even multiples with an additive analysis $2+3+p$ where only one of the added numbers is 2, we observe that this is a general property of the even multiples, since with the acceptance of the generalized Goldbach's conjecture we have for every $m \geq 2$,
$$3(2m) = p+q+u.$$
One of the odd number additive primes must be even number, i.e., equal to 2, since for otherwise their sum would have been odd number and not even $3(2m)$. Let such a number be $u$. Then
$$3(2m) = 2+p+q.$$
This implies that for the next odd multiple we have
$$3(2m+1) = 3(2m)+3 = 5+p+q$$
so that it admits an analysis of sum of three primes. This particular implies that in order to prove the generalized conjecture for 3, it suffices to show the analysis of every multiple of 3 as the sum of three prime numbers. We deduce that this equals with the Goldbach's conjecture. Therefore it also remains as an open problem waiting for its answer.

### 5.2. Case $a = 4$

In general, the analysis of the multiples $4m$ of 4 remains as an open problem.

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