α-GENERALIZED-CONVERGENCE THEORY OF L-FUZZY NETS AND ITS APPLICATIONS

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Abstract. The convergence theory not only is an significantly basic theory of fuzzy topology and fuzzy analysis but also has wide applications in fuzzy inference and some other aspects. In this paper, we introduce the concept of α-generalized-remote-neighborhood of fuzzy points and establish the Moore-Smith α-generalized-convergence theory of L-fuzzy nets. Then, we introduce and study the concept of L-fuzzy α-generalized-irresolute mappings and L-fuzzy α-generalized compactness. Also we discuss the applications of α-generalized-convergence of L-fuzzy nets.

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1. Introduction

The usual notion of a set was generalized with the introduction of fuzzy sets by Zadeh in the classical paper [15] of 1965. Since then many authors have expansively developed the theory of fuzzy sets and its applications to several sectors of both pure and applied sciences, such as [6], [10]-[14]. As it is known now that the traditional neighborhood method is not effective any longer in fuzzy topology, in order to overcome this deficiency Pu and Liu introduced the concepts of the fuzzy point and the Q-neighborhood and established a systematic Moore-Smith convergence theory of fuzzy nets [10]. It paved a new way for the study of the fuzzy topology. Later on, Wang introduced the concept of remote-neighborhood systems [13], this concept is an abstraction of the concept of neighborhood in point set topology and the concept of Q-neighborhood in fuzzy topology. Q-neighborhood and remote-neighborhood can be used in wide aspect [3], [8]-[13].

In this paper, we introduce the concept of α-generalized-remote-neighborhood of fuzzy points with the concept of remote-neighborhood. In Section 3, using the concepts α-generalized fuzzy closed sets and L-fuzzy α-generalized-remote-neighborhood, we establish the Moore-Smith α-generalized-convergence theory of L-fuzzy nets. In Section 4, we discuss the applications of α-generalized-convergence of L-fuzzy nets.

2. Preliminaries

Throughout this paper, $L = L(\leq, \vee, \wedge, ^{\prime})$ will denote a fuzzy lattice, i.e., a completely distributive lattice with a smallest element 0 and largest element 1 ($0 \neq 1$) and with an order reversing involution $a \to a^{'} (a \in L)$. Let X be a nonempty crisp set, and we shall denote by L^X the lattice of all L-subsets of X and if $A \subseteq X$ by χ_A the characteristic function of A. An element p of L is called prime iff $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leq p$ then $a \leq p$ or $b \leq p$ [14]. The set of all prime elements of L will be denoted by $Pr(L)$. An element α of L is called union-irreducible or coprime iff whenever $a, b \in L$ with $\alpha \leq a \vee b$ then $\alpha \leq a$ or $\alpha \leq b$ [14]. The set of all non-zero union-irreducible elements of L will be denoted by $M(L)$. It is obvious that $p \in pr(L)$ iff $p^{'} \in M(L)$. We denote $M^*(L^X) = \{x_\alpha : x \in X \text{ and } \alpha \in M(L)\}$.

For the definition of a fuzzy point x_α we follow Pu and Liu [10]. When the support and value of a fuzzy point are trivial, we use briefly the symbols e to denote fuzzy point. A fuzzy point $x_\alpha \in A$, where A is an L-fuzzy set in X, iff $\alpha \leq A(x)$. The constant L-fuzzy sets taking on the values 0 and 1 on X are designated by 0_X and 1_X, respectively. An L-fuzzy net $S = \{S(n), n \in D\}$ is a function $S : D \to \xi$ where D is a directed set with order relation $\geq$ and ξ the collection of all the fuzzy points in X [14]. A net S is called an $\alpha-$net ($\alpha \in M(L)$) if for each $\lambda \in \beta^{'}(\alpha)$ (where $\beta^{'}(\alpha)$ denotes the union of all minimal sets relative to α), there is $n_0 \in D$ such that $V(S(n)) \geq \lambda$ whenever $n \geq n_0$, where $V(S(n))$ is the height of point $S(n)$.

Definition 2.1. Let L be a fuzzy lattice, X be a nonempty crisp set and $\delta \subseteq L^X$. An L-fuzzy topology is a family δ of L-subsets of X which satisfies the following conditions:

(a) $0, 1 \in \delta$,
(b) If $A, B \in \delta$, then $A \wedge B \in \delta$,
(c) If $A_i \in \delta$ for each $i \in I$, then $\vee_{i \in I} A_i \in \delta$.

δ is called an L-fuzzy topology on X, and the pair (L^X, δ) is an L-fuzzy topological space, or L-fts for short. Every member of δ is called L-fuzzy open set.

Example 2.2. Let $L = \{0, 1\}$, then $L^X \cong 2^X$ and (L^X, δ) is just the general topological space.

Definition 2.3. Let $L = [0, 1]$, then (L^X, δ) is called fuzzy topological space.

Remark 2.4. From the definitions above, we know that L-fuzzy topological space is the generalization of both general topological space and fuzzy topological space.

Example 2.5. Let $X = [0,1]$ and define fuzzy sets on X as:

$$\mu_1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2; \\ 2x-1, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1/4; \\ 2-4x, & \text{if } 1/4 \leq x \leq 1/2; \\ 0, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

$$\mu_3(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4; \\ (4x-1)/3, & \text{if } 1/4 \leq x \leq 1. \end{cases}$$

Put $\tau = \{0, 1, \mu_3\}$ and $\sigma = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2\}$. Then (X, τ) and (X, σ) are both fuzzy topological spaces and hence L-fuzzy topological spaces.

Let (L^X, δ) be an L-fuzzy topological space (briefly, L-fts), e be a fuzzy point and P an L-fuzzy closed set in (L^X, δ). Then P is called a remote-neighborhood of e, if $e \not\in P$. The set of all remote-neighborhoods of e will be denoted by $\eta(e)$. A^o, A^- and $A^{'}$ will denote the interior, closure and complement of the L-fuzzy set A in X, respectively. For definitions and results not explained in this paper, the reader is referred to [10], [13] assuming them to be well known.

Example 2.6. Let $x_1, x_2, ...$ be a sequence in a set X. Then it is a net with an index set $D = \{1, 2, ...\}$. So the concept of a net is a generalization of the concept of a sequence.

Definition 2.7. Let L_1 and L_2 be fuzzy lattices. A mapping $f : L_1 \to L_2$ is called an order-homomorphism (briefly, OH) if the following conditions hold:

(1) $f(0) = 0$.

(2) $f(\vee A_i) = \vee f(A_i)$ for $\{A_i\} \subset L_1$.

(3) $f^{-1}(B^{'}) = (f^{-1}(B))'$ for each $B \in L_2$.

In general topological spaces, generalized closed sets were introduced by Norman Levine [5]. G. Balasubramanian and P. Sundaram extended this definition to L-topological spaces ($L = [0,1]$) [2].

Definition 2.8. (G. Balasubramanian and P. Sundaram [2]) Let (L^X, δ) be an L-fts and $f \in L^X$. Then f is called generalized fuzzy closed (in short gfc) iff $cl(f) \leq \mu$ whenever $f \leq \mu$ and μ is L-fuzzy open. An L-set λ is called generalized fuzzy open (in short gfo) iff $1-\lambda$ is gfc. It can be proved that λ is gfo iff $\mu \leq Int(\lambda)$ whenever $\mu \leq \lambda$ and μ is L-fuzzy closed. And the union of any two gf-closed sets is also a gf-closed set.

And in general topological spaces, α-generalized closed sets were introduced by H. Maki, R. Devi, and K. Balachandran in [7]. M.E. El-Shafei and A. Zakari extended this definition to L-topological spaces and studied its basic properties in [3].

Definition 2.9. (M.E. El-Shafei and A. Zakari [3]) Let (L^X, δ) be an L-fts and $f \in L^X$. Then f is called α-generalized fuzzy closed (in short α-g-closed) iff $cl_\alpha(f) \leq \mu$ whenever $f \leq \mu$ and μ is L-fuzzy open. It's easy to see that a finite union of α-generalized fuzzy closed sets is always α-generalized fuzzy closed set. And the complement of a α-g-closed fuzzy set is called α-g-open.

Proposition 2.10. [3] *Every generalized fuzzy closed set is α-g-closed.*

Let (L^X, δ) be an L-fts, and A an L-set of (L^X, δ). Then

$$A_\triangle = \cup\{B : B \in \alpha GO(L^X),\ B \leq A\},\ \ A_\sim = \cap\{B : B \in \alpha GC(L^X),\ A \leq B\}$$

are called the α-generalized-interior and α-generalized-closure of A, respectively. $\alpha GO(L^X)$ and $\alpha GC(L^X)$ will always denote the family of α-g-open sets and the family of α-g-closed sets of an L-fts (L^X, δ), respectively.

α-Generalized-convergence of L-fuzzy nets

Definition 3.1. Let (L^X, δ) be an L-fts, x_α be a fuzzy point and $P \in \alpha GC(L^X)$. P is called an L-fuzzy α-generalized-remote-neighborhood, or briefly, αGC-RN of x_α, if $x_\alpha \notin P$. The set of all αGC-RNs of x_α will be denoted by ζ_{x_α}.

Definition 3.2. Let A an L-set of an L-fts (L^X, δ). Then a fuzzy point x_α is called a α-g-adherence point of A if $A \not\leq P$ for each $P \in \zeta_{x_\alpha}$. If x_α is a α-g-adherence point of A and $x_\alpha \notin A$, or $x_\alpha \in A$ and for each fuzzy point x_μ satisfying $x_\alpha \leq x_\mu \in A$ we have $A \not\leq x_\mu \vee P$, then x_α is called a α-g-accumulation point of A. The union of all α-g-accumulation points of A will be called the α-G-derived set of A and denoted $A^{d(\alpha-G)}$.

Remark 3.3. In general topological space (or in mathematical analysis), a point x is an adherence point of a subset A iff every neighborhood of x intersects A; A point x is an accumulation point of a subset A iff every neighborhood of x contains points of A other than x.

In general topological space, P is a closed remote-neighborhood of a point x iff P' is an open neighborhood of x. Then every closed remote-neighborhood P of x does not contain A equivalent to every open neighborhood P' of x intersected A. So the concept of adherence point in L-fts is a generalization of the adherence point in general topological space (or in analysis). Similarly, the concept of accumulation point in L-fts is a generalization of the accumulation point in general topological space (or in analysis).

Definition 3.4. Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then

(1) e is said to be a α-g-limit point of S (or S α-g-converges to e; in symbols, $S \to e(\alpha)$), if for each $P \in \zeta(e)$, $S(n) \notin P$ is eventually true (i.e. if there exists $n_0 \in D$ such that for every $n \in D$, $n \geq n_0$, always possess $S(n) \notin P$).

(2) e is said to be a α-g-cluster point of S (or S α-g-accumulates to e; in symbols, $S \infty e(\alpha)$), if for each $P \in \zeta(e)$, $S(n) \notin P$ is frequently true (i.e. if for every $n_0 \in D$, there always exist $n \in D$, $n \geq n_0$, such that $S(n) \notin P$).

The union of all α-g-limit points and all α-g-cluster points of S will be denoted by α-g-lim S and α-g-ad S, respectively. Obviously, α-g-lim $S \leq \alpha$-g-ad S. One can readily check the following proposition.

Proposition 3.5. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then the following statements are valid:*

(1) *If $S = \{S(n) : n \in D\} \to e(\alpha)$, $T = \{T(n) : n \in D\}$ is an L-fuzzy net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$ holds. Then $T = \{T(n) : n \in D\} \to e(\alpha)$.*

(2) *If $S = \{S(n) : n \in D\} \infty e(\alpha)$, $T = \{T(n) : n \in D\}$ is an L-fuzzy net with the same domain as S and for each $n \in D$, $T(n) \geq S(n)$ holds. Then $T = \{T(n) : n \in D\} \infty e(\alpha)$.*

(3) *If $S=\{S(n) : n \in D\} \to e(\alpha)$ and $d \leq e$. Then $S = \{S(n) : n \in D\} \to d(\alpha)$.*

(4) *If $S = \{S(n) : n \in D\} \infty e(\alpha)$ and $d \leq e$. Then $S = \{S(n) : n \in D\} \infty d(\alpha)$.*

Theorem 3.6. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then:*

(1) *$S \to e(\alpha)$ iff $e \in \alpha$-g-lim S.*

(2) *$S \infty e(\alpha)$ iff $e \in \alpha$-g-ad S.*

Proof. (1) $\Rightarrow$ Suppose that $S \to e(\alpha)$, then by the Definition 3.4, e is said to be a α-g-limit point of S. And α-g-lim S is the union of all α-g-limit points of S, then we have $e \in \alpha$-g-lim S.

$\Leftarrow$ Suppose that $e \in \alpha$-g-lim S and $P \in \zeta(e)$. Then $e \notin P$, and so α-g-lim $S \not\leq P$. By the definition of α-g-lim S, there must exist a α-g-limit point d of S such that $d \notin P$, i.e., $P \in \zeta(d)$. Hence, S is eventually not in P, i.e., $S \to e(\alpha)$.

(2) $\Rightarrow$ Suppose that $S \infty e(\alpha)$, then by Definition 3.4, e is said to be a α-g-cluster point of S. And α-g-ad S is the union of all α-g-cluster points of S, then we have $e \in \alpha$-g-ad S.

$\Leftarrow$ Suppose that $e \in \alpha$-g-ad S and $P \in \zeta(e)$. Then $e \notin P$, and so α-g-ad $S \not\leq P$. By the definition of α-g-ad S, there must exist a α-g-cluster point d of S such that $d \notin P$, i.e., $P \in \zeta(d)$. Hence, $S \notin P$ is frequently true, i.e., $S \infty e(\alpha)$.

Theorem 3.7. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then α-g-lim S and α-g-ad S are α-g-closed.*

Proof. Let $e \in (\alpha\text{-g-lim } S)_\sim$. Then α-g-lim $S \not\leq P$ for each $P \in \zeta(e)$. Hence there exists $d \in M^*(L^X)$ such that $d \in \alpha$-g-lim S and $d \notin P$. Then $P \in \zeta(d)$. By Theorem 3.6 (1), $S \to d(\alpha)$, i.e., $S(n) \notin P$ is eventually true. Thus, $e \in \alpha$-g-lim S. This implies that α-g-lim S is α-g-closed. Similarly, α-g-ad S is α-g-closed.

Theorem 3.8. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $A \in L^X$.*

(1) *If there exists in A an L-fuzzy net $S = \{S(n) : n \in D\}$ such that $S \infty e(\alpha)$, then e is a α-g-adherence point of A.*

(2) *If e is a α-g-adherence point of A, then there exists in A an L-fuzzy net $S = \{S(n) : n \in D\}$ such that $S \to e(\alpha)$.*

Proof. (1) Let $S \infty e(\alpha)$ and $S(n) \in A$ for each $n \in D$. Then for each $P \in \zeta(e)$. $A \not\leq P$ because of the fact that $S(n) \notin P$ is frequently true. Hence, e is a α-g-adherence point of A.

(2) If e is a α-g-adherence point of A, then for each $P \in \zeta(e)$ there exists a point $S(P)$ such that $S(P) \leq A$ and $S(P) \not\leq P$. Define $S = \{S(P), P \in \zeta(e)\}$, then S is an L-fuzzy net in A because of the fact that $\zeta(e)$ is a directed set in which the order is defined by inclusion. Clearly, $S \to e(\alpha)$.

Definition 3.9. Let $S = \{S(n) : n \in D\}$ and $T = \{T(m) : m \in E\}$ be two nets in L^X. Call T the subnet of S, if there exists a mapping $N : E \to D$ such that

(1) $T = SN$;

(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

Theorem 3.10. *Let (L^X, δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. Then S has a subnet T such that $T \to e(\alpha)$ iff $S \infty e(\alpha)$.*

Proof. Suppose that $T = \{T(m) : m \in E\}$ is a subnet of S, $T \to e(\alpha)$, $P \in \zeta(e)$ and $n_0 \in D$. By the definition of subnet, there exists a mapping $N : E \to D$ and $m_0 \in E$ such that $N(m) \geq n_0(N(m) \in D)$ when $m \geq m_0(m \in E)$. Since T α-g-converges to e, there is $m_1 \in E$. When $m \geq m_1(m \in E)$, $T(m) \notin P$. Because E is a directed set, there exists $m_2 \in E$ such that $m_2 \geq m_0$ and $m_2 \geq m_1$. Hence, $T(m_2) \notin P$ and $N(m_2) \geq n_0$. Let $n = N(m_2)$. Then $S(n) = S(N(m_2)) = T(m_2) \notin P$ and $n \geq n_0$. This means that $S(n) \notin P$ is frequently true. Thus $S \infty e(\alpha)$.

Conversely, suppose that $S \infty e(\alpha)$. Then for each $P \in \zeta(e)$ and each $n \in D$, there exists $N(P, n) \in D$ such that $N(P, n) \geq n$ and $S(N(P, n)) \notin P$. Let $E = \{(N(P, n), P) : P \in \zeta(e), n \in D\}$, and define $(N(P_1, n_1), P_1) \leq (N(P_2, n_2), P_2)$ iff $n_1 \leq n_2$ and $P_1 \leq P_2$. Thus E is a directed set because:

(a) For each $(N(P,n),P)$, since $n \in D$ and D is a directed set, we have $n \leq n$. Also, since $P \in \zeta(e)$ and $\zeta(e)$ is a directed set, we have $P \leq P$. Hence we have $n \leq n$ and $P \leq P$ which equivalent that $(N(P,n),P) \leq (N(P,n),P)$. Thus the relation $\leq$ is reflexive on E.

(b) Let $(N(P_1,n_1),P_1)$, $(N(P_2,n_2),P_2)$ and $(N(P_3,n_3),P_3)$ belong to E with $(N(P_1,n_1),P_1) \leq (N(P_2,n_2),P_2)$ and $(N(P_2,n_2),P_2) \leq (N(P_3,n_3),P_3)$. Thus we have $n_1 \leq n_2$, $P_1 \leq P_2$ and $n_2 \leq n_3$ and $P_2 \leq P_3$. Since D and $\zeta(e)$ are directed sets, we get $n_1 \leq n_3$ and $P_1 \leq P_3$ which equivalent that $(N(P_1,n_1),P_1) \leq (N(P_3,n_3),P_3)$. Thus the relation $\leq$ is transitive on E.

(c) Let $(N(P_1,n_1),P_1)$ and $(N(P_2,n_2),P_2)$ belong to E. Since $n_1,n_2 \in D$ and D is a directed set, there is $n \in D$ such that $n_1 \leq n$ and $n_2 \leq n$. Also, since $P_1,P_2 \in \zeta(e)$, we have $P = P_1 \vee P_2 \in \zeta(e)$ and $P_1 \leq P$, $P_2 \leq P$. Hence there exists $(N(P,n),P) \in E$ with $(N(P_1,n_1),P_1) \leq (N(P,n),P)$ and $(N(P_2,n_2),P_2) \leq (N(P,n),P)$.

Hence $(E,\leq)$ is a direct set. Let $T(N(P,n),P) = S(N(P,n))$. Then T is a subnet of S and $T \to e(\alpha)$.

Theorem 3.11. *Let (L^X,δ) be an L-fts, $e \in M^*(L^X)$ and $S = \{S(n) : n \in D\}$ an L-fuzzy net in L^X. If T is a subnet of S, then:*

(1) *If $S \to e(\alpha)$, then $T \to e(\alpha)$.*

(2) *If $T \infty e(\alpha)$, then $S \infty e(\alpha)$.*

(3) *α-g-lim $S \leq \alpha$-g-lim T.*

(4) *α-g-ad $T \leq \alpha$-g-ad S.*

Proof. (1) Suppose $T = \{T(m) : m \in E\}$ is a subnet of S, $S \to e(\alpha)$ and $P \in \zeta(e)$, then $S(n) \notin P$ is eventually true. From the definition of subnet, there exists a mapping $N : E \to D$ and for every $m \in E$, there exists $n \in D$ such that $T(m) = S(N(m)) = S(n)$. That is to say, every element of the net T is actually the element of the net S. So $T(m) \notin P$ is eventually true. Thus we have $T \to e(\alpha)$.

(2) Suppose that $T = \{T(m) : m \in E\}$ is a subnet of S, $T \infty e(\alpha)$, $P \in \zeta(e)$ and $n_0 \in D$. By the definition of subnet, there exists a mapping $N : E \to D$ and $m_0 \in E$ such that $N(m) \geq n_0(N(m) \in D)$ when $m \geq m_0(m \in E)$. Since T α-g-accumulates to e, for $m_0 \in E$ there is $m_1 \in E$. When $m_1 \geq m_0(m_1 \in E)$, $T(m_1) \notin P$. Let $n = N(m_1)$. Then $S(n) = S(N(m_1)) = T(m_1) \notin P$ and $n \geq n_0$. This means that $S(n) \notin P$ is frequently true. Thus $S \infty e(\alpha)$.

(3) By Theorem 3.6, $S \to e(\alpha)$ means $e \in \alpha$-g-lim S and $T \to e(\alpha)$ means $e \in \alpha$-g-lim T. Thus by (1), we have α-g-lim $S \leq \alpha$-g-lim T.

(4) By Theorem 3.6, $S \infty e(\alpha)$ means $e \in \alpha$-g-ad S and $T \infty e(\alpha)$ means $e \in \alpha$-g-ad T. Thus by (2), we have α-g-ad $T \leq \alpha$-g-ad S.

4. Applications

Definition 4.1. An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is said to be α-g-irresolute if $f^{-1}(B) \in \alpha GO(L_1^X)$ for each $B \in \alpha GO(L_2^Y)$.

Theorem 4.2. *For an OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ the following are equivalent:*

(1) *f is α-g-irresolute.*

(2) *$f^{-1}(B) \in \alpha GC(L_1^X)$ for each $B \in \alpha GC(L_2^Y)$.*

(3) *$(f^{-1}(B))_\sim \leq f^{-1}(B_\sim)$ for each $B \in L_2^Y$.*

Proof. (1)⇒(2): f is α-g-irresolute if $f^{-1}(A) \in \alpha GO(L_1^X)$ for each $A \in \alpha GO(L_2^Y)$. For each $B \in \alpha GC(L_2^Y)$, $B' \in \alpha GO(L_2^Y)$. So we have $(f^{-1}(B))' = f^{-1}(B') \in \alpha GO(L_1^X)$. This shows $f^{-1}(B) \in \alpha GC(L_1^X)$.

(2)⇒(1): For each $A \in \alpha GO(L_2^Y)$, $A' \in \alpha GC(L_2^Y)$. Then by (2) we have $(f^{-1}(A))' = f^{-1}(A') \in \alpha GC(L_1^X)$. This shows $f^{-1}(A) \in \alpha GO(L_1^X)$. Hence by Definition 4.1, f is α-g-irresolute.

(2)⇒(3): For each $B \in L_2^Y$, $B_\sim \in \alpha GC(L_2^Y)$. Then by (2) we have $f^{-1}(B_\sim) \in \alpha GC(L_1^X)$. And $B \leq B_\sim$ implies $f^{-1}(B) \leq f^{-1}(B_\sim)$. From the definition of α-generalized-closure we have $(f^{-1}(B))_\sim \leq f^{-1}(B_\sim)$.

(3)⇒(1): Let $B \in \alpha GC(L_2^Y)$, then $B = B_\sim$. By (3) we have $f^{-1}(B) \leq (f^{-1}(B))_\sim \leq f^{-1}(B_\sim) = f^{-1}(B)$, i.e., $f^{-1}(B) = (f^{-1}(B))_\sim$. Hence $f^{-1}(B) \in \alpha GC(L_2^X)$ and consequently, f is α-g-irresolute.

Definition 4.3. An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is said to be α-g-irresolute at a point $e \in M^*(L_1^X)$ if $(f^{-1}(P))_\sim \in \zeta_1(e)$ for each $P \in \zeta_2(f(e))$, where $\zeta_1(e)$ and $\zeta_2(f(e))$ denote the set of all αGC-RNs of e and $f(e)$, respectively.

Theorem 4.4. *An OH $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ is α-g-irresolute iff f is α-g-irresolute for each point $e \in M^*(L_1^X)$.*

Proof. Suppose that f is α-g-irresolute and $e \in M^*(L_1^X)$. Then $f^{-1}(P)$ is α-g-closed for each $P \in \zeta_2(f(e))$. Clearly, $e \notin f^{-1}(P)$. Hence $f^{-1}(P) = (f^{-1}(P))_\sim \in \zeta_1(e)$ and so f is α-g-irresolute at e.

Conversely, suppose that f is α-g-irresolute for each $e \in M^*(L_1^X)$ and $P \in \alpha GC(L_2^Y)$. We may assume that $f^{-1}(P) \neq 1_X$ and suppose that $e \notin f^{-1}(P)$. Then $f(e) \notin P$ and so $P \in \zeta_2(f(e))$. Hence, $(f^{-1}(P))_\sim \in \zeta_1(e)$, i.e., $e \notin f^{-1}(P)$ implies that $e \notin (f^{-1}(P))_\sim$ or $(f^{-1}(P))_\sim \leq f^{-1}(P)$. Thus, $f^{-1}(P)$ is α-g-closed in (L_1^X, δ), i.e., f is α-g-irresolute.

Now we discuss the applications of α-g-convergence of L-fuzzy nets.

Theorem 4.5. *Let $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ be α-g-irresolute at $e \in M^*(L_1^X)$ and S an L-fuzzy net in L_1^X. If $S \to e(\alpha)$ we have $f(S)$ α-g-converges to $f(e)$ where $f(S) = \{f(S(n)),\ n \in D\}$ is an L-fuzzy net in L_2^Y.*

Proof. Suppose that f is α-g-irresolute at $e \in M^*(L_1^X)$ and $S \to e(\alpha)$. Let $P \in \zeta_2(f(e))$. Then S is eventually not in $(f^{-1}(P))_\sim \in \zeta_1(e)$, and hence $f(S)$ is eventually not in P, i.e., $f(S) \to f(e)(\alpha)$.

Theorem 4.6. *Let f :$(L_1^X, \delta) \to (L_2^Y, \tau)$ be α-g-irresolute. Then for each L-fuzzy net S in L_1^X we have $f(\alpha$-g-lim $S) \leq \alpha$-g-lim $f(S)$.*

Proof. Suppose that $e \in M^*(L_1^X)$, S is an L-fuzzy net in L_1^X and $f(e) \in f(\alpha$-g-lim $S)$. Then $e \in \alpha$-g-lim S. By Theorem 3.6 we have $S \to e(\alpha)$. Since f is α-g-irresolute, we have $f(S) \to f(e)(\alpha)$ based on Theorems 4.4 and 4.5. And by Theorem 3.6 we have $f(e) \in \alpha$-g-lim $f(S)$. Thus, $f(\alpha$-g-lim $S) \leq \alpha$-g-lim $f(S)$.

Theorem 4.7. *Let $f : (L_1^X, \delta) \to (L_2^Y, \tau)$ be α-g-irresolute. Then for each L-fuzzy net T in L_2^Y we have α-g-lim $f^{-1}(T) \leq f^{-1}(\alpha$-g-lim $T)$.*

Proof. Let $T = \{T(n) : n \in D\}$ be an L-fuzzy net in L_2^Y. Then $f^{-1}(T) = \{f^{-1}(T(n)) : n \in D\}$ an L-fuzzy net in L_1^X. Since f is α-g-irresolute, according to Theorem 4.6 we have $f(\alpha$-g-lim $f^{-1}(T)) \leq \alpha$-g-lim $f(f^{-1}(T)) \leq \alpha$-g-lim T. Hence, α-g-lim $f^{-1}(T) \leq f^{-1}(\alpha$-g-lim $T)$.

Definition 4.8. (Aygün [1]) Let (L^X, δ) be an L-fts and $g \in L^X$, $r \in L$.

(1) A collection $\mu = \{f_i\}_{i \in J}$ of L-subsets is called an r-level cover of g iff $(\vee_{i \in J} f_i)(x) \nleq r$ for all $x \in X$ with $g(x) \geq r'$. If each f_i is open then μ is called an r-level open cover of g. If g is the whole space 1_X, then μ is called an r-level cover of 1_X iff $(\vee_{i \in J} f_i)(x) \nleq r$ for all $x \in X$.

(2) An r-level cover $\mu = \{f_i\}_{i \in J}$ of g is said to have a finite r-level subcover if there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \nleq r$ for all $x \in X$ with $g(x) \geq r'$.

Definition 4.9. (Kudri [4]) Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be compact iff for every prime $p \in L$ and every collection $\{f_i\}_{i \in J}$ of open L-subsets with $(\vee_{i \in J} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exists a finite subset F of J such that $(\vee_{i \in F} f_i)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, i.e. every p-level open cover of g has a finite p-level subcover, where $p \in pr(L)$. If g is the whole space, then the L-tfs (L^X, δ) is called compact.

Definition 4.10. Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is called α-g-compact iff every p-level cover of g consisting of α-g-open L-subsets has a finite p-level subcover, where $p \in pr(L)$. If g is the whole space, then we say that the L-fts (X, δ) is α-g-compact.

Theorem 4.11. *Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be α-g-compact if and only if for every $\alpha \in M(L)$ and every collection $(f_i)_{i \in J}$ of α-g-closed L-fuzzy sets with $(\wedge_{i \in J} f_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, there exists a finite subset F of J with $(\wedge_{i \in F} f_i)(x) \ngeq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, i.e., L-fuzzy points $x_\alpha \in M(L^X)$ such that $x_\alpha \leq g$.*

Proof. This follows immediately from Definition 4.10 and the duality of p and α.

Definition 4.12. Let (L^X, δ) be an L-fts , $x_\alpha \in M^*(L^X)$ and $S = (S_m)_{m\in D}$ be a net. x_α is called a α-g-cluster of S iff for each α-g-closed L-subset f with $f(x) \not\geq \alpha$ and for all $n \in D$, there is $m \in D$ such that $m \geq n$ and $S_m \not\leq f$, i.e., $h(S_m) \not\leq f(SuppS_m)$.

Theorem 4.13. *Let (L^X, δ) be an L-fts and $g \in L^X$. The L-fuzzy subset g is said to be α-g-compact if and only if for every constant $\alpha-$net $(S_m)_{m\in D}$ contained in $g(S_m \leq g$ for every $m \in D)$ has a α-g-cluster point with height $\alpha, x_\alpha \in M^*(L^X)$, contained in $g(x_\alpha \leq g$ for each $\alpha \in M(L))$.*

Proof. *Necessity:* Let $\alpha \in M(L)$ and $S = (S_m)_{m\in D}$ be a constant α-net in g without any α-g-cluster point with height α in g. Then for each $x \in X$ with $g(x) \geq \alpha$, x_α is not a α-g-cluster point of S, i.e., there are $n_x \in D$ and a α-g-closed L-subset f_x with $f_x(x) \not\geq \alpha$ and $S_m \leq f_x$ for each $m \geq n_x$. Let $x^1, ..., x^k$ be elements of X with $g(x^i) \geq \alpha$ for each $i \in \{1, ..., k\}$. Then there are $n_{x1}, ..., n_{xk} \in D$ and α-g-closed L-subset f_{xi} with $f_{xi}(x^i) \not\geq \alpha$ and $S_m \leq f_{xi}$ for each $m \geq n_{xi}$ and for each $i \in \{1, ..., k\}$. Since D is a directed set, there is $n_o \in D$ such that $n_o \geq n_{xi}$ for each $i \in \{1, ..., k\}$ and $S_m \leq f_{xi}$ for $i \in \{1, ..., k\}$ and each $m \geq n_o$. Now, consider the family $\mu = \{f_x\}_{x\in X}$ with $g(x) \geq \alpha$. Then $(\wedge_{f_x\in\mu} f_x)(y) \not\geq \alpha$ for all $y \in X$ with $g(y) \geq \alpha$ because $f_y(y) \not\geq \alpha$. We also have that for any finite subfamily $\nu = \{f_{x1}, ..., f_{xk}\}$ of μ, there is $y \in X$ with $g(y) \geq \alpha$ and $(\wedge_{i=1}^k f_{xi})(y) \geq \alpha$ since $S_m \leq \wedge_{i=1}^k f_{xi}$ for each $m \geq n_o$ because $S_m \leq f_{xi}$ for each $i \in \{1, ..., k\}$ and for each $m \geq n_o$. Hence, by Theorem 4.11, g is not α-g-compact.

Sufficiency: Suppose that g is not α-g-compact. Then, by Theorem 4.11, there exist $\alpha \in M(L)$ and a collection $\mu = \{f_i\}_{i\in J}$ of α-g-closed L-subsets with $(\wedge_{i\in J} f_i)(x) \not\geq \alpha$ for all $x \in X$ with $g(x) \geq \alpha$, but for any finite subfamily ν of μ there is $x \in X$ with $g(x) \geq \alpha$ and $(\wedge_{f\in\nu} f)(x) \geq \alpha$. Consider the family of all finite subsets of μ, $2^{(\mu)}$, with the order $\nu_1 \leq \nu_2$ iff $\nu_1 \subseteq \nu_2$. Then $2^{(\mu)}$ is a directed set. So, writing x_α as S_ν for every $\nu \in 2^{(\mu)}$, $(S_\nu)_{\nu\in 2^{(\mu)}}$ is a constant α-net in g because the height of S_ν for all $\nu \in 2^{(\mu)}$ is α and $S_\nu \leq g$ for all $\nu \in 2^{(\mu)}$, i.e., $g(x) \geq \alpha$. $(S_\nu)_{\nu\in 2^{(\mu)}}$ also satisfies the condition that for each α-g-closed L-subset $f_i \in \nu$ we have $x_\alpha = S_\nu \leq f_i$. Let $y \in X$ with $g(y) \geq \alpha$. Then $(\wedge_{i\in J} f_i)(y) \not\geq \alpha$, i.e., there exists $j \in J$ with $f_j(y) \not\geq \alpha$. Let $\nu_0 = \{f_j\}$. So, for any $\nu \geq \nu_0$, $S_\nu \leq \wedge_{f_i\in\nu} f_i \leq \wedge_{f_i\in\nu_0} f_i = f_j$. Thus, we get a α-g-closed L-subset f_j with $f_j(y) \not\geq \alpha$ and $\nu_0 \in 2^{(\mu)}$ such that for any $\nu \geq \nu_0$, $S_\nu \leq f_j$. That means that $y_\alpha \in M^*(L^X)$ is not a α-g-cluster point of $(S_\nu)_{\nu\in 2^{(\mu)}}$ for all $y \in X$ with $g(y) \geq \alpha$. Hence, the constant α-net $(S_\nu)_{\nu\in 2^{(\mu)}}$ has no α-g-cluster point in g with height α.

Corollary 4.14. *An L-fts is α-g-compact iff every constant α-net in (L^X, δ) has a α-g-cluster point with height α, where $\alpha \in M(L)$.*

5. Conclusion and future research

The theory of fuzzy lattices is one of the most important branches in fuzzy systems. The theory of α-g-closed sets, α-g-convergence of L-fuzzy nets and α-g-irresolute functions and α-g-compact which presented in this paper by using molecules and remoted neighborhoods are very significant tools to studying the theory of L-fuzzy topological spaces. There is still a lot of results for future investigations, for example the consideration of this theory on topological molecular lattices [14] will lead to some interesting research from the view point of fuzzy mathematics.
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References

[1] Aygün, H., *α-compactness in L-fuzzy topological spaces*, Fuzzy Sets Syst., 116 (2000), 317–324.

[2] Balasubramanian, Sundaram, P., *On some generalizations of fuzzy continuous functions*, Fuzzy Sets and Systems., 86 (1997), 93–100.

[3] El-Shafei, M.E., Zakari, A., *θ-Generalized closed sets in fuzzy topological spaces*, The Arabian Journal for Science and Engineering, 31 (2006), 197–206.

[4] Kudri, S.R.T., *Compactness in L-fuzzy topological spaces*, Fuzzy Sets Syst., 67 (1994), 329–336.

[5] Levine, N., *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, 19 (1970), 89–96.

[6] Lowen, R., *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl., 56 (1976), 621–633.

[7] Maki, H., Devi, R. and Balachandran, K., *Associated topologies of generalized α-closed sets and α-generalized closed sets*, Mem. Fac. Sci. Kochi Univ., Ser. A Math., 15 (1994), 51–63.

[8] Nouh, A.A., Seyam, N., *HC-Convergence theory of L-fuzzy nets and L-fuzzy ideals and its applications*, The Arabian Journal for Science and Engineering, 30 (2005), 317–334.

[9] Nouh, A.A., *On convergence theory in fuzzy topological spaces and its applications*, Czechoslovak Mathematical Journal, 55 (2005), 295–316.

[10] Pu, B.M., Liu, Y.M., *Fuzzy topology* I. *Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., 76 (1980), 571–599.

[11] Seenivasan, V., Balasubramanian, G., *On upper and lower α-irresolute fuzzy multifunctions*, Italian Journal of Pure and Applied Mathematics, 23 (2008), 11–24.

[12] Uma, M.K., Roja, E., Balasubramanian, G., *On pairwise fuzzy pre-basically and pre-extremally disconnected spaces*, Italian Journal of Pure and Applied Mathematics, 23 (2008), 75–84.

[13] Wang, G.J., *A new fuzzy compactness defined by fuzzy nets*, J. Math. Anal. Appl., 94 (1983), 59–67.

[14] Wang, G.J., *Theory of Topological Molecular Lattices*, Fuzzy Sets and Systems, 47 (1992), 351–376.

[15] Zadeh, L.A., *Fuzzy sets*, Inform Cont., 8 (1965), 338–353.

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