INTERVAL-VALUED INTUITIONISTIC FUZZY SUBSEMIMODULES WITH (S,T)-NORMS

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Abstract. On the basis of the concept of the interval valued intuitionistic fuzzy sets introduced by K. Atanassov, the notion of interval valued intuitionistic fuzzy subsemimodule of a semimodule with respect to t-norm T and s-norm S is given and the characteristic properties are described. The homomorphic image and inverse image are investigated. In particular, by the help of the congruence relations on semimodules, new interval valued intuitionistic (S,T)-fuzzy subsemimodules are constructed.

Keywords: semimodule, subsemimodule, interval valued intuitionistic (S,T)-fuzzy subsemimodule.

1. Introduction

After the introduction of fuzzy sets by Zadeh [14], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [1], [2], [3]. In 1975, Zadeh [15] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers. Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation, in a plastic products company and in medicine (see the book [3]).

The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc. Also the notion of fuzzy submodules in modules and semimodules (in different views) have seriously studied by many mathematicians ([11], [12]). Recently, some researchers are trying to present new views of fuzzy algebraic structures as intuitionistic fuzzy algebraic structures ([10], [16]). In algebra, we notice that the subsemimodules of semimodules play a crucial role in the structure theory, but they do not in general coincide with the usual submodules, for this reason, their usage is somewhat limited when we try to obtain some analogous module theorems for semimodules. Indeed, many results in modules apparently have no analogous in semimodules by using only submodules. In this paper we introduce the notion of interval valued intuitionistic fuzzy subsemimodules of a semimodule with respect to t-norm T and s-norm S. Then we characterize all of them base on special kind of levels $\mathfrak{U}(\widetilde{M}_{\mathcal{A}};[t,s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}};[t,s])$, which is a generalization of classic level subsets. At the following the behaviour of this structure under homomorphisms is investigated. In particular, by the help of the congruence relations on semimodules, we construct new interval valued intuitionistic (S,T)-fuzzy subsemimodules on semimodule of quotient.

2. Preliminaries and notations

Let $\mathcal{SR}$ be a semiring. A *left $\mathcal{SR}$-semimodule* is a commutative semigroup $\mathcal{SM}$ which we have a function $\mathcal{SR}\times\mathcal{SM}\longrightarrow\mathcal{SM}$, denote by $(r,m)\mapsto rm$ and called *scalar multiplication*, which satisfies the following conditions for all $r,r'\in\mathcal{SR}$ and $m,m'\in\mathcal{SM}$:

(1) $(rr')m = r(r'm)$;

(2) $r(m+m') = rm + rm'$;

(3) $(r+r')m = rm + rm'$.

Right semimodules over $\mathcal{SR}$ are defined in an analogous manner. A *semimodule* is both left and right semimodule (see [6]).

A non-empty subset $\mathcal{SN}$ of a left $\mathcal{SR}$-semimodule $\mathcal{SM}$ is a *subsemimodule* of $\mathcal{SM}$ if and only if $\mathcal{SN}$ is closed under addition and scalar multiplication.

An equivalence relation ρ on a semigroup $(\mathcal{SM},.)$ is said to be a *congruence relation*, if for all $x,y,z\in\mathcal{SM}$, $x\rho y$ implies $(xz)\rho(yz)$, where by $x\rho y$ we mean $(x,y)\in\rho$. Also by $\mathcal{SM}/\rho$ we mean the set of all equivalence classes with respect to ρ, or $\mathcal{SM}/\rho=\{\rho(x) : x\in\mathcal{SM}\}$ (see [6]). Also an equivalence relation θ on a semiring $(\mathcal{SR},+,.)$ is said to be a congruence relation, if for all $x,y,z\in\mathcal{SR}$, $x\theta y$ implies $(x+z)\theta(y+z)$ and $(xz)\theta(yz)$ (see [6]).

By an *interval number* $\widetilde{a}$ we mean ([15]) an interval $[a^-,a^+]$, where $0\le a^-\le a^+\le 1$. The set of all interval number is denoted by $D[0,1]$. The interval $[a,a]$ is identified whit the number $a\in[0,1]$.

For interval numbers $\widetilde{a}_i=[a_i^-,a_i^+]\in D[0,1], i\in I$, we define (see [3] and [15])

$$\inf\widetilde{a}_i=\left[\bigwedge_{i\in I}a_i^-,\bigwedge_{i\in I}a_i^+\right],\qquad \sup\widetilde{a}_i=\left[\bigvee_{i\in I}a_i^-,\bigvee_{i\in I}a_i^+\right]$$

and put

(1) $\widetilde{a_1} \leq \widetilde{a_2} \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,

(2) $\widetilde{a_1} = \widetilde{a_2} \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(3) $\widetilde{a_1} < \widetilde{a_2} \Longleftrightarrow \widetilde{a_1} \leq \widetilde{a_2}$ and $\widetilde{a_1} \neq \widetilde{a_2}$,

(4) $k\tilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

It is clear that $(D[0,1], \leq, \bigvee, \bigwedge)$ is a complete lattice whit $0 = [0,0]$ as the least element and $1 = [1,1]$ as the greatest element.

By an *interval number fuzzy set* F on X we mean ([15]) the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) : x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subset of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\mu_F(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \mu_F(x)) : x \in X\}$, where $\mu_F : X \longrightarrow D[0,1]$.

As it is well-known, the function $\delta : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *t-norm* (resp. *s-norm*) if δ satisfied the conditions:

(i) $\delta(x,1) = x$ (resp. $\delta(x,0) = x$),

(ii) $\delta(x,y) = \delta(y,x)$,

(iii) $\delta(\delta(x,y),z) = \delta(x,\delta(y,z))$,

(iv) $\delta(x,u) \leq \delta(x,w)$, for all $x,y,z,u,w \in [0,1]$, where $u \leq w$.

A t-norm (resp. s-norm) δ is called an *idempotent t-norm* if $\delta(x,x) = x$, for all $x \in [0,1]$, (see [17]).

If δ is an idempotent t-norm (s-norm), then the mapping

$$\Delta : D[0,1] \times D[0,1] \longrightarrow D[0,1]$$

defined by

$$\Delta(\widetilde{a}_1, \widetilde{a}_2) = [\delta(a_1^-, a_2^-), \delta(a_1^+, a_2^+)]$$

is, as it is not difficult to verify, an idempotent t-norm (s-norm, respectively) and is called an *idempotent interval t-norm (s-norm, respectively).*

According to Atanassov ([1], [2], [3]), *an interval valued intuitionistic fuzzy* set on X is defined as an object of the form

$$\mathcal{A} = \{(x, \widetilde{M}_{\mathcal{A}}(x), \widetilde{N}_{\mathcal{A}}(x)) : x \in X\},$$

where $\widetilde{M}_{\mathcal{A}}(x)$ and $\widetilde{N}_{\mathcal{A}}(x)$ are interval valued fuzzy sets on X such that

$$0 \leq \sup \widetilde{M}_{\mathcal{A}}(x) + \sup \widetilde{N}_{\mathcal{A}}(x) \leq 1 \ \text{ for all } x \in X.$$

For the sake of simplicity, in the following such interval valued intuitionistic fuzzy sets will be denoted by $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$.

3. Interval valued intuitionistic (s,t)-fuzzy subsemimodules of semimodules

In what follows, let $\mathcal{SM}$ denote a $\mathcal{SR}$-semimodule unless otherwise specified.

Definition 3.1. An interval valued intuitionistic fuzzy set $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ of $\mathcal{SM}$ is called an interval valued intuitionistic (S,T)-fuzzy left subsemimodule of $\mathcal{SM}$ if for all $x, y \in \mathcal{SM}$ and $r \in \mathcal{SR}$ we have

(1) $\widetilde{M}_{\mathcal{A}}(x+y) \geq T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y))$, $\widetilde{N}_{\mathcal{A}}(x+y) \leq S(\widetilde{N}_{\mathcal{A}}(x), \widetilde{N}_{\mathcal{A}}(y))$,

(2) $\widetilde{M}_{\mathcal{A}}(rx) \geq \widetilde{M}_{\mathcal{A}}(x)$, $\widetilde{N}_{\mathcal{A}}(rx) \leq \widetilde{N}_{\mathcal{A}}(x)$.

Similarly, we define an interval valued intuitionistic (S,T)-fuzzy right subsemimodule. An interval valued intuitionistic (S,T)-fuzzy subsemimodule is both interval valued intuitionistic (S,T)-fuzzy left and right subsemimodule.

Example. A commutative semigroup $(\mathcal{SM},+)$ is a $\mathbb{N}$-semimodule with the function $\mathbb{N}\times\mathcal{SM} \longrightarrow \mathcal{SM}$ defined by $(i,m) \mapsto im = m$. Let $\mathcal{SN}$ be a subsemimodule of $\mathcal{SM}$ and let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.8, 0.9], & \text{if } x \in \mathcal{SN} \\ [0.1, 0.2], & \text{if } x \notin \mathcal{SN} \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0.2, 0.3], & \text{if } x \in \mathcal{SN} \\ [0.7, 0.8], & \text{if } x \notin \mathcal{SN} \end{cases}$$

it can easily to be checked that $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}$.

Example. $\mathbb{Z}^- = \{0, -1, -2, -3, ...\}$ with the rule $\mathbb{N} \times \mathbb{Z}^- \longrightarrow \mathbb{Z}^-$ defined by $(n,a) \mapsto na$ is a $\mathbb{N}$-semimodule. Let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.9, 1], & \text{if } x = 0, -2, -4, -6, ... \\ [0, 0.1], & \text{if } x = -1, -3, -5, ... \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0, 0.1], & \text{if } x = 0, -2, -4, -6, ... \\ [0.9, 1], & \text{if } x = -1, -3, -5, ... \end{cases}$$

it is easy to calculate that $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathbb{Z}^-$. Now let

$$\widetilde{M}_{\mathcal{A}}(x) = \begin{cases} [0.1, 0.2], & \text{if } \ x = 0, -2, -4, -6, ... \\ [0.8, 0.9], & \text{if } \ x = -1, -3, -5, ... \end{cases}$$

$$\widetilde{N}_{\mathcal{A}}(x) = \begin{cases} [0.8, 0.9], & \text{if } \ x = 0, -2, -4, -6, ... \\ [0.1, 0.2], & \text{if } \ x = -1, -3, -5, ... \end{cases}$$

since $\widetilde{M}_{\mathcal{A}}(2 \times (-3)) = \widetilde{M}_{\mathcal{A}}(-6) = [0.1, 0.2]$ and $\widetilde{M}_{\mathcal{A}}(-3) = [0.8, 0.9]$ and so, $\widetilde{M}_{\mathcal{A}}(2 \times (-3)) \not\geq \widetilde{M}_{\mathcal{A}}(-3)$, therefore $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is not an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathbb{Z}^-$.

With any interval valued intuitionistic fuzzy set $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ of $\mathcal{SM}$ there are connected two levels:

$$\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s]) = \{x \in \mathcal{SM} : \widetilde{M}_{\mathcal{A}}(x) \geq [t, s]\},$$

$$\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t, s]) = \{x \in \mathcal{SM} : \widetilde{N}_{\mathcal{A}}(x) \leq [t, s]\}.$$

Theorem 3.2. *Let T and S be idempotent intervals t-norm and s-norm respectively. Then $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy submodule if and only if for all $t, s \in [0,1], t \leq s, \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t, s])$ are subsemimodules of $\mathcal{SM}$.*

Proof. Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic (S,T)-fuzzy subsemimodle of $\mathcal{SM}$. Then for every $x, y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ we have $\widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$ and $\widetilde{M}_{\mathcal{A}}(y) \geq [t, s]$. Hence $T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y)) \geq T([t, s], [t, s]) = [t, s]$, and so $\widetilde{M}_{\mathcal{A}}(x+y) \geq [t, s]$. Therefore $x+y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$. If $r \in \mathcal{SR}$ and $x \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$, then $\widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$. On the other hand $\widetilde{M}_{\mathcal{A}}(rx) \geq \widetilde{M}_{\mathcal{A}}(x) \geq [t, s]$. Therefore

$$rx \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s]).$$

This proves that $\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ is a subsemimodule of $\mathcal{SM}$.

Conversely, assume that for every $[t, s] \in D[0,1]$ any non-empty $\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t, s])$ is a subsemimodule of $\mathcal{SM}$. If $[t_0, s_0] = T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y))$ for some $x, y \in \mathcal{SM}$, then $x, y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_0, s_0])$ and so $x + y \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_0, s_0])$. Therefore

$$\widetilde{M}_{\mathcal{A}}(x+y) \geq [t_0, s_0] = T(\widetilde{M}_{\mathcal{A}}(x), \widetilde{M}_{\mathcal{A}}(y)).$$

Also if $[t_1, s_1] = \widetilde{M}_{\mathcal{A}}(x)$, for some $x \in \mathcal{SM}$, then $x \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_1, s_1])$, and so $rx \in \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t_1, s_1])$, for every $r \in \mathcal{SR}$, hence $\widetilde{M}_{\mathcal{A}}(rx) \geq [t_1, s_1] = \widetilde{M}_{\mathcal{A}}(x)$. This proves that $\widetilde{M}_{\mathcal{A}}$ is an interval valued intuitionistic left T-fuzzy subsemimodule of $\mathcal{SM}$. The proof of $\widetilde{M}_{\mathcal{A}}$ is an interval valued intuitionistic right T-fuzzy subsemimodule of $\mathcal{SM}$ is similar. Analogously, we can show that $\widetilde{N}_{\mathcal{A}}$ is an interval valued intuitionistic S-fuzzy ideal of $\mathcal{SM}$. Therefore $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule. ∎

Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic fuzzy set of $\mathcal{SR}$ and let $t, s, t', s' \in [0,1]$ such that $t \leq s$ and $t' \leq s'$. Put

$$\mathcal{M}^{[t,s]}_{[t',s']} = \{x \in \mathcal{SM} : \widetilde{M}_{\mathcal{A}}(x) \geq [t,s],\ \widetilde{N}_{\mathcal{A}}(x) \leq [t',s']\}.$$

Clearly,

$$\mathcal{M}^{[t,s]}_{[t',s']} = \mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t,s]) \bigcap \mathfrak{L}(\widetilde{M}_{\mathcal{A}}; [t',s']).$$

Corollary 3.3. *Let T and S be idempotent intervals t-norm and s-norm respectively. Then $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}$ if and only if for all $t, s, t', s' \in [0,1], t \leq s,\ t' \leq s'$, $\mathcal{M}^{[t,s]}_{[t',s']}$ is a subsemimodule of $\mathcal{SM}$.*

Proof. It is immediately followed by Theorem 3.2. ■

Definition 3.4. Let $f : X \to Y$ be a mapping and $\mathcal{A}=(\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ and $\mathcal{B}=(\widetilde{M}_{\mathcal{B}}, \widetilde{N}_{\mathcal{B}})$ interval valued intuitionistic sets X and Y, respectively. Then the image of $f[\mathcal{A}] = (f(\widetilde{M}_{\mathcal{A}}), f(\widetilde{N}_{\mathcal{A}}))$ of $\mathcal{A}$ is the interval valued intuitionistic fuzzy set of Y defined by

$$f(\widetilde{M}_{\mathcal{A}})(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \widetilde{M}_{\mathcal{A}}(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ [0,0] & \text{otherwise} \end{cases}$$

$$f(\widetilde{N}_{\mathcal{A}})(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} \widetilde{N}_{\mathcal{A}}(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ [1,1] & \text{otherwise} \end{cases}$$

for all $y \in Y$.

The inverse image $f^{-1}(\mathcal{B})$ of $\mathcal{B}$ is an interval valued intuitionistic fuzzy set defined by

$$f^{-1}(\widetilde{M}_{\mathcal{B}})(x) = \widetilde{M}_{f^{-1}(\mathcal{B})}(x) = \widetilde{M}_{\mathcal{B}}(f(x)),$$

$$f^{-1}(\widetilde{N}_{\mathcal{B}})(x) = \widetilde{N}_{f^{-1}(\mathcal{B})}(x) = \widetilde{N}_{\mathcal{B}}(f(x))$$

for all $x \in X$.

Definition 3.5. Let $\mathcal{SM}$ and $\mathcal{SN}$ be two semimodules over a semiring $\mathcal{SR}$. A mapping $f : \mathcal{SM} \longrightarrow \mathcal{SN}$ is called a homomorphism if for all $x, y \in \mathcal{SM}$ and $r \in \mathcal{SR}$ we have $f(x+y) = f(x) + f(y)$ and $f(r.x) = r.f(x)$.

Lemma 3.6. *Let $\mathcal{SM}_1$ and $\mathcal{SM}_2$ be two semimodules over a semiring $\mathcal{SR}$ and $f : \mathcal{SM}_1 \longrightarrow \mathcal{SM}_2$ an epimorphism.*

(i) *If $\mathcal{SN}_1$ is a subsemimodule of $\mathcal{SM}_1$, then $f(\mathcal{SN}_1)$ is a subsemimodule of $\mathcal{SM}_2$.*

(ii) *If $\mathcal{SN}_2$ is a subsemimodule of $\mathcal{SM}_2$, then $f^{-1}(\mathcal{SN}_2)$ is a subsemimodule of $\mathcal{SM}_1$.*

Proof. Straightforward. ∎

Theorem 3.7. *Let $\mathcal{SM}_1$ and $\mathcal{SM}_2$ be two subsemimodules, and $f : \mathcal{SM}_1 \to \mathcal{SM}_2$ an epimorphism and T and S idempotent intervals t-norm and s-norm respectively.*

(i) *If $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$, then the image $f[\mathcal{A}]$ of $\mathcal{A}$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$.*

(ii) *If $\mathcal{B} = (\widetilde{M}_{\mathcal{B}}, \widetilde{N}_{\mathcal{B}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$, then the inverse image $f^{-1}(\mathcal{B}) = (f^{-1}(\widetilde{M}_{\mathcal{B}}), f^{-1}(\widetilde{N}_{\mathcal{B}}))$ of $\mathcal{B}$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$.*

Proof. (i) Let $\mathcal{A}=(\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_1$. By Theorem 3.2, $\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t,s])$ and $\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t,s])$ are subsemimodules of $\mathcal{SM}_1$ for every $[t,s]\in D[0,1]$. Therefore, by Lemma 3.6, $f(\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t,s]))$ and $f(\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t,s]))$ are subsemimodules of $\mathcal{SM}_2$. But

$$\mathfrak{U}(f(\widetilde{M}_{\mathcal{A}}); [t,s]) = f(\mathfrak{U}(\widetilde{M}_{\mathcal{A}}; [t,s])) \text{ and } \mathfrak{L}(f(\widetilde{N}_{\mathcal{A}}); [t,s]) = f(\mathfrak{L}(\widetilde{N}_{\mathcal{A}}; [t,s])),$$

so, $\mathfrak{U}(f(\widetilde{M}_{\mathcal{A}}); [t,s])$ and $\mathfrak{L}(f(\widetilde{N}_{\mathcal{A}}); [t,s])$ are subsemimodules of $\mathcal{SM}_2$. Therefore $f[A]$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}_2$.

(ii) For any $x, y \in \mathcal{SM}_1$, we have

$$\begin{aligned} \widetilde{M}_{f^{-1}(\mathcal{B})}(x+y) &= \widetilde{M}_{\mathcal{B}}(f(x+y)) \geq T(\widetilde{M}_{\mathcal{B}}(f(x)), \widetilde{M}_{\mathcal{B}}(f(y))) \\ &= T(\widetilde{M}_{f^{-1}(\mathcal{B})}(x), \widetilde{M}_{f^{-1}(\mathcal{B})}(y)). \end{aligned}$$

Also, if $x \in \mathcal{SM}_1$ and $r \in SR$, we have

$$\widetilde{M}_{f^{-1}(\mathcal{B})}(r.x) = \widetilde{M}_{\mathcal{B}}(f(r.x)) = \widetilde{M}_{\mathcal{B}}(r.f(x)) \geq \widetilde{M}_{\mathcal{B}}(f(x)) = \widetilde{M}_{f^{-1}(\mathcal{B})}(x).$$

This completes the proof that $\widetilde{M}_{f^{-1}(\mathcal{B})}$ is an interval valued T-fuzzy subsemimodule of $\mathcal{SR}_1$. Similarly we can prove $\widetilde{N}_{f^{-1}(\mathcal{B})}$ is an interval valued S-fuzzy subsemimodule of $\mathcal{SR}_1$. Similarly $f^{-1}(\mathcal{B}) = (f^{-1}(\widetilde{M}_{f^{-1}(\mathcal{B})}), f^{-1}(\widetilde{N}_{f^{-1}(\mathcal{B})}))$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SR}_1$. ∎

Let γ be a congruence relation on $\mathcal{SM}$ and θ a congruence relation on $\mathcal{SR}$. Then it is easy to verify that $\mathcal{SM}/\gamma$ is a semimodule over semiring $\mathcal{SR}/\theta$, by the rule $\odot : \mathcal{SM}/\gamma \times \mathcal{SR}/\theta \longrightarrow \mathcal{SM}/\gamma$ define by $\gamma(x) \odot \theta(r) = \gamma(x.r)$.

Definition 3.8. Let $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ be an interval valued intuitionistic fuzzy set. The intuitionistic fuzzy set $\mathcal{A}/\gamma = (\widetilde{M}_{\mathcal{A}/\gamma}, \widetilde{N}_{\mathcal{A}/\gamma})$ is defined as a pair of maps

$$\begin{cases} \widetilde{M}_{\mathcal{A}/\gamma} : \mathcal{SR}/\gamma \longrightarrow D[0,1] \\ \widetilde{N}_{\mathcal{A}/\gamma} : \mathcal{SR}/\gamma \longrightarrow D[0,1] \end{cases}$$

Such that $\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)) = \sup\limits_{a\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(a)$ and $\widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)) = \inf\limits_{a\in\gamma(x)} \widetilde{N}_{\mathcal{A}}(a)$.

Theorem 3.9. *Let $\mathcal{SM}$ be a semimodule over $\mathcal{SR}$. If $\mathcal{A} = (\widetilde{M}_{\mathcal{A}}, \widetilde{N}_{\mathcal{A}})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of $\mathcal{SM}$, then $\mathcal{A}/\gamma = (\widetilde{M}_{\mathcal{A}/\gamma}, \widetilde{N}_{\mathcal{A}/\gamma})$ is an interval valued intuitionistic (S,T)-fuzzy subsemimodule of semimodule $\mathcal{SM}/\gamma$ over $\mathcal{SR}/\theta$.*

Proof. Let $\gamma(x), \gamma(y) \in \mathcal{SM}/\gamma$, we have

$$\begin{aligned}
T(\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(y))) &= T\left(\sup_{a\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(a), \sup_{b\in\gamma(y)} \widetilde{M}_{\mathcal{A}}(b)\right) \\
&= \sup_{a\in\gamma(x),\ b\in\gamma(y)} T(\widetilde{M}_{\mathcal{A}}(a), \widetilde{M}_{\mathcal{A}}(b)) \\
&\leq \sup_{a\in\gamma(x),\ b\in\gamma(y)} \widetilde{M}_{\mathcal{A}}(a+b) \\
&\leq \sup_{a\in\gamma(x),\ b\in\gamma(y)} \left(\sup_{t\in\gamma(a+b)} \widetilde{M}_{\mathcal{A}}(t)\right) \\
&= \sup_{a\in\gamma(x),\ b\in\gamma(y)} \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b)) \\
&= \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b)),
\end{aligned}$$

for all $a \in \gamma(x), b \in \gamma(y)$. On the other hand, we have

$$\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a+b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(a) \oplus \gamma(b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x+y)).$$

So,

$$T(\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(y))) \leq \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)).$$

The proof of the inequality

$$S(\widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)), \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(y))) \geq \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x) \oplus \gamma(y)),$$

is similar.

To prove the second condition, let $\gamma(x) \in \mathcal{SM}/\gamma$ and $\theta(r) \in \mathcal{SR}/\theta$, then for every $b \in \gamma(x)$ we have

$$\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) = \widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(b)) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(r.b)).$$

On the other hand

$$\widetilde{M}_{\mathcal{A}/\gamma}(\gamma(r.b)) = \sup_{t\in\gamma(r.b)} \widetilde{M}_{\mathcal{A}}(t) \geq \widetilde{M}_{\mathcal{A}}(r.b) \geq \widetilde{M}_{\mathcal{A}}(b),$$

and so for every $b \in \gamma(x)$, we have $\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \geq \widetilde{M}_{\mathcal{A}}(b)$. Hence

$$\widetilde{M}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \geq \sup_{b\in\gamma(x)} \widetilde{M}_{\mathcal{A}}(b) = \widetilde{M}_{\mathcal{A}/\gamma}(\gamma(x)).$$

Similarly, we can obtain

$$\widetilde{N}_{\mathcal{A}/\gamma}(\theta(r) \odot \gamma(x)) \leq \widetilde{N}_{\mathcal{A}/\gamma}(\gamma(x)).$$

This completes the proof.

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