FUZZY MINIMAL STRUCTURES AND FUZZY MINIMAL SUBSPACES

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Abstract. At the present paper, the notions of induced fuzzy minimal structures, fuzzy minimal subspaces and relatively fuzzy minimal continuous functions are introduced and studied.

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1. Introduction

After the discovery of the fuzzy sets by Zadeh [17], many attempts have been made to extend various branches of mathematics to the fuzzy setting. Fuzzy topological spaces as a very natural generalization of topological spaces were first put forward in the literature by Chang [7] in 1968. He studied a number of the basic concepts including interior and closure of a fuzzy set, fuzzy continuous mapping and fuzzy compactness. Many authors used Chang's definition in many direction to obtain some results which are compatible with results in general topology. In 1976, Lowen [9] suggested an alternative and more natural definition for achieving more results which are compatible to the general case in topology. For example with Chang's definition, constant functions between fuzzy topological spaces are not necessarily fuzzy continuous but in Lowen's sense all of the constant functions are fuzzy continuous. In 1985, Sostak [16] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology.

The concept of minimal structure and minimal spaces, as a generalization of topology and topological spaces were introduced in [11]. Further results about minimal spaces can be found in [2], [5], [10] and [15]. Recently, Alimohammady and Roohi [3], [4] introduced and studied the notions of fuzzy minimal structures and fuzzy minimal spaces.

This paper is organized as follows. In Section 2, some preparatory definitions and results about fuzzy sets which are used in other sections are given. Section 3 is devoted to reviewing some basic definitions and results on fuzzy minimal structures and fuzzy minimal spaces. Also, some new results and an example are investigated. Finally, in Section 4, the concepts of induced fuzzy minimal structures, fuzzy minimal subspaces and relatively fuzzy minimal continuous functions are introduced and studied.

2. Preliminaries

To ease understanding of the material incorporated in this paper, we recall some basic definitions and results. For details on the following notions, we refer to [1], [3], [4], [12], [13] and references therein.

A *fuzzy set* in(on) a universe set X is a function with domain X and values in $I = [0, 1]$. The class of all fuzzy sets on X will be denoted by I^X and symbols A,B,... is used for fuzzy sets on X. For two fuzzy sets A and B in X, we say that A is *contained in* B provided $A(x) \leq B(x)$ for all $x \in X$. The *complement* of A is denoted by A^c and is defined by $A^c(x) = 1 - A(x)$. 01_X is called *empty fuzzy set* where 1_X is the characteristic function on X. A family τ of fuzzy sets in X is called a *fuzzy topology* for X if

(a) $\alpha 1_X \in \tau$ for each $\alpha \in I$,

(b) $A \wedge B \in \tau$, where $A, B \in \tau$ and

(c) $\bigvee_{\alpha \in \mathcal{A}} A_\alpha \in \tau$ whenever, $A_\alpha \in \tau$ for all α in $\mathcal{A}$. The pair (X, τ) is called a *fuzzy topological space* [9]. Every member of τ is called *fuzzy open set* and its complement is called *fuzzy closed sets* [9]. In a fuzzy topological space X, the *interior* and the *closure* of a fuzzy set A (denoted by $Int(A)$ and $Cl(A)$ respectively) are defined by

$$Int(A) = \bigvee \{U \ : \ U \leq A, \ U \text{ is fuzzy open set}\} \text{ and}$$
$$Cl(A) = \bigwedge \{F \ : \ A \leq F, \ F \text{ is fuzzy closed set}\}.$$

Let f be a function from X to Y. It is a fuzzy function defined by

$$f(A)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(\{y\})} A(x) & f^{-1}(\{y\}) \neq \emptyset \\ 0 & f^{-1}(\{y\}) = \emptyset, \end{cases}$$

for all y in Y, where A is an arbitrary fuzzy set in X and also $f^{-1}(B) = Bof$ for any fuzzy set B in Y [17].

A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ $(0 < \lambda \leq 1)$, we denote this fuzzy point by x_λ, where the point x is called its *support* [12], [13].

3. Fuzzy minimal spaces

Definition 3.1. A family $\mathcal{M}$ of fuzzy sets in X is said to be a

(a) *fuzzy minimal structure in Lowen sense* on X if $\lambda 1_X \in \mathcal{M}$ for any $\lambda \in I$, where $I = [0, 1]$ ([4]).

(b) *fuzzy minimal structure in Chang sense* on X if $\lambda 1_X \in \mathcal{M}$ for any $\lambda \in \{0, 1\}$ ([3]).

In these cases, $(X, \mathcal{M})$ is called a *fuzzy minimal space in Lowen sense* (resp. Chang sense).

In the rest of this paper, fuzzy minimal structure is used for fuzzy minimal structure in Lowen sense.

A fuzzy set $A \in I^X$ is said to be *fuzzy m-open* if $A \in \mathcal{M}$ and also $B \in I^X$ is called a *fuzzy m-closed* set if $B^c \in \mathcal{M}$. Let

$$m - Int(A) = \bigvee\{U : U \leq A, U \in \mathcal{M}\} \quad \text{and} \tag{3.1}$$

$$m - Cl(A) = \bigwedge\{F : A \leq F, F^c \in \mathcal{M}\}. \tag{3.2}$$

Proposition 3.2. [4], [6] *For any two fuzzy sets A and B,*

(a) *m-$Int(A) \leq A$ and m-$Int(A) = A$ if A is a fuzzy m-open set.*

(b) *$A \leq m$-$Cl(A)$ and $A = m$-$Cl(A)$ if A is a fuzzy m-closed set.*

(c) *m-$Int(A) \leq m$-$Int(B)$ and m-$Cl(A) \leq m$-$Cl(B)$ if $A \leq B$.*

(d) *m-$Int(A \wedge B) = (m$-$Int(A)) \wedge (m$-$Int(B))$ and $(m$-$Int(A)) \vee (m$-$Int(B)) \leq m$-$Int(A \vee B)$.*

(e) *m-$Cl(A \vee B) = (m$-$Cl(A)) \vee (m$-$Cl(B))$ and m-$Cl(A \wedge B) \leq (m$-$Cl(A)) \wedge (m$-$Cl(B))$.*

(f) *m-$Int(m$-$Int(A)) = m$-$Int(A)$ and m-$Cl(m$-$Cl(B)) = m$-$Cl(B)$.*

(g) *$(m$-$Cl(A))^c = m$-$Int(A^c)$ and $(m$-$Int(A))^c = m$-$Cl(A^c)$.*

Definition 3.3. [4] A fuzzy minimal space $(X, \mathcal{M})$ enjoys the *property* U if arbitrary union of fuzzy m-open sets is fuzzy m-open. Also, we say that $(X, \mathcal{M})$ has the *property* I, if any finite intersection of fuzzy m-open sets is fuzzy m-open.

Proposition 3.4. [1] *For a fuzzy minimal structure $\mathcal{M}$ on a set X, the following are equivalent.*

(a) $(X, \mathcal{M})$ *has the property* U.

(b) *If* m-$Int(A) = A$, *then* $A \in \mathcal{M}$.

(c) *If* m-$Cl(B) = B$, *then* $B^c \in \mathcal{M}$.

Definition 3.5. [4] Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two fuzzy minimal spaces. We say that a fuzzy function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is *fuzzy minimal continuous* (briefly fuzzy m-continuous) if $f^{-1}(B) \in \mathcal{M}$, for any $B \in \mathcal{N}$.

Theorem 3.6. [3], [4] *Suppose* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *are fuzzy minimal spaces. Then*

(a) *the identity map* $id_X : (X, \mathcal{M}) \to (X, \mathcal{M})$ *is fuzzy m-continuous,*

(b) $id_X : (X, \mathcal{M}) \to (X, \mathcal{N})$ *is fuzzy m-continuous if and only if* $\mathcal{N} \subseteq \mathcal{M}$,

(c) *Any constant function* $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *is fuzzy m-continuous.*

Theorem 3.7. [3], [4] *Consider the following properties for a fuzzy function* $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *between two fuzzy minimal spaces.*

(a) f *is a fuzzy m-continuous function.*

(b) $f^{-1}(B)$ *is a fuzzy m-closed set for each fuzzy m-closed set* $B \in I^Y$.

(c) m-$Cl(f^{-1}(B)) \leq f^{-1}(m$-$Cl(B))$ *for each* $B \in I^Y$.

(d) $f(m$-$Cl(A)) \leq m$-$Cl(f(A))$ *for any* $A \in I^X$.

(e) $f^{-1}(m$-$Int(B)) \leq m$-$Int(f^{-1}(B))$ *for each* $B \in I^Y$.

Then (a)⇔(b)⇒(c)⇔(d)⇔(e). *Moreover, if* $(X, \mathcal{M})$ *satisfies in property* U *then all above statements are equivalent.*

Example 3.8. Let $X=\{x, y\}$, $\mathcal{M}=\{\lambda 1_X : \lambda \in I\} \cup \{x_1\}$ and $\mathcal{N}=\{\lambda 1_X : \lambda \in I\} \cup \{x_{\frac{1}{2}}\}$. It follows from part (b) of Theorem 3.6 that $Id_X : (X, \mathcal{M}) \longrightarrow (X, \mathcal{N})$ is not fuzzy m-continuous. Let m_1-Cl and m_2-Cl are denoted for fuzzy minimal closure in $(X, \mathcal{M})$ and $(X, \mathcal{N})$ respectively. Then, for any fuzzy set B in X with $B(x) = s$ and $B(y) = t$, it follows from (3.2) that m_1-Cl(f^{-1}(B)) $\leq$ f^{-1}(m$_2$-Cl(B)) for each $B \in I^X$.

In [4], for a family of fuzzy functions, authors achieved a weakest fuzzy minimal structure for which all members of it are fuzzy m-continuous. As a consequence, *fuzzy product minimal structure* for an arbitrary family $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in \mathcal{A}\}$ of fuzzy minimal spaces are introduced. In fact, fuzzy product minimal structure on $X = \prod_{\alpha\in A} X_\alpha$ is the weakest fuzzy minimal structure on X (denoted by $\mathcal{M} = \prod_{\alpha\in\mathcal{A}} \mathcal{M}_\alpha$) such that for each $\alpha \in \mathcal{A}$ the canonical projection $\pi_\beta : X \longrightarrow X_\beta$ is fuzzy m-continuous. It should be noticed that fuzzy product minimal structure for two fuzzy minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ is the family of fuzzy sets

$$\mathcal{M} \times \mathcal{N} = \{1_X \times V : V \in \mathcal{N}\} \cup \{U \times 1_Y : U \in \mathcal{M}\}.$$

Similarly, one can verify that fuzzy product minimal structure of $\{(X_j, \mathcal{M}_j) : j = 1, 2, ..., n\}$ is

$$\prod_{j=1}^{n} \mathcal{M}_j = \bigcup_{j=1}^{n} \left\{ \prod_{l=1}^{n} F_l : F_l = \left\{ \begin{array}{ll} 1_{X_l} & l \neq j \\ U_j & l = j, \text{ where, } U_j \in \mathcal{M}_j \end{array} \right. \right\}. \tag{3.3}$$

We use $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$ instead of $\prod\limits_{j=1}^{n} \mathcal{M}_i$ and specially $\mathcal{M}_1 \times \mathcal{M}_2$ instead of $\prod\limits_{j=1}^{2} \mathcal{M}_i$.

Theorem 3.9. [4] *Suppose $\{(X_\alpha, M_\alpha) : \alpha \in \mathcal{A}\}$ is a family of fuzzy minimal spaces. Equip X by the fuzzy product minimal structure $\mathcal{M}$ generated by $\{\pi_\alpha : \alpha \in \mathcal{A}\}$. Then f is fuzzy m-continuous function if and only if $\pi_\alpha o f$ is fuzzy m-continuous for all $\alpha \in \mathcal{A}$, where $f : (Y, \mathcal{N}) \to (X, \mathcal{M})$ is a mapping.*

Theorem 3.10. *Suppose $(X, \mathcal{M})$ is a fuzzy minimal space, $\{(Y_\alpha, \mathcal{M}_\alpha) : \alpha \in \mathcal{A}\}$ is a family of fuzzy minimal spaces and also suppose that $(Y, \mathcal{N})$ is the fuzzy product minimal space of this family. Then for all $\alpha \in \mathcal{A}$, $f_\alpha : (X, \mathcal{M}) \longrightarrow (Y_\alpha, \mathcal{M}_\alpha)$ is fuzzy m-continuous if and only if $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$, defined by $f(x) = (f_\alpha(x))_\alpha$, is fuzzy m-continuous.*

Proof. Clearly, $\pi_\alpha o f = f_\alpha$ and hence $\pi_\alpha o f$ is fuzzy m-continuous for all $\alpha \in \mathcal{A}$. That f is fuzzy m-continuous follows from Theorem 3.9.

Theorem 3.11. *Suppose $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are fuzzy minimal spaces. Then for each $y_0 \in Y$ the mapping $i_{y_o} : (X, \mathcal{M}) \longrightarrow (X \times Y, \mathcal{M} \times \mathcal{N})$ defined by $i_{y_0}(x) = (x, y_0)$ is fuzzy m-continuous.*

Proof. By part (c) of Theorem 3.6, the mapping $C_{y_0} : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ defined by $C_{y_0}(x) = y_0$, for all $x \in X$, is fuzzy m-continuous also part (a) of Theorem 3.6 implies that the identity mapping $id_X : (X, \mathcal{M}) \longrightarrow (X, \mathcal{M})$ is fuzzy m-continuous too. That i_{y_o} is fuzzy m-continuous follows from Theorem 3.10.

Similarly, one can deduce the following result.

Theorem 3.12. *Suppose* $(X, \mathcal{M})$, $(Y, \mathcal{N})$ *are fuzzy minimal spaces. Then for each* $x_0 \in X$ *the mapping* $i_{x_o} : (Y, \mathcal{N}) \longrightarrow (X \times Y, \mathcal{M} \times \mathcal{N})$ *defined by* $i_{x_0}(y) = (x_0, y)$ *is fuzzy* m*-continuous.*

4. Fuzzy minimal subspaces

Definition 4.1. Let A be a fuzzy set in X and $\mathcal{M}$ be a fuzzy minimal space on X. Then $\mathcal{M}_A = \{U \wedge A : U \in \mathcal{M}\}$ is called an *induced fuzzy minimal structure on* A and $(A, \mathcal{M}_A)$ is called *fuzzy minimal subspace* of $(X, \mathcal{M})$.

Proposition 4.2. *Suppose* $(A, \mathcal{M}_A)$ *is a fuzzy minimal subspace of fuzzy minimal space. If*

(a) $(X, \mathcal{M})$ *has the property* U*, then* $(A, \mathcal{M}_A)$ *has this property,*

(b) $(X, \mathcal{M})$ *has the property* I*, then* $(A, \mathcal{M}_A)$ *has this property too.*

Proof. Consider a family $\{V_\alpha : \alpha \in \mathcal{A}\}$ of fuzzy sets in $\mathcal{M}_A$, then there exists a family $\{U_\alpha : \alpha \in \mathcal{A}\}$ of fuzzy m-open sets in $(X, \mathcal{M})$ such that $V_\alpha = U_\alpha \wedge A$ for all $\alpha \in \mathcal{A}$. Therefore,

$$\bigvee_{\alpha \in \mathcal{A}} V_\alpha = \bigvee_{\alpha \in \mathcal{A}} (U_\alpha \wedge A) = \left(\bigvee_{\alpha \in \mathcal{A}} U_\alpha \right) \wedge A.$$

$\bigvee_{\alpha \in \mathcal{A}} V_\alpha \in \mathcal{M}_A$ follows from the fact that $(X, \mathcal{M})$ has the property U, which proves (a). The proof of (b) is similarly.

Definition 4.3. Suppose $(A, \mathcal{M}_A)$ and $(B, \mathcal{N}_A)$ are fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ respectively. Also, suppose that $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is a mapping. We say that f is a mapping from $(A, \mathcal{M}_A)$ into $(B, \mathcal{N}_A)$ if $f(A) \leq B$.

Definition 4.4. Suppose $(A, \mathcal{M}_A)$ and $(B, \mathcal{N}_A)$ are fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ respectively. The mapping f from $(A, \mathcal{M}_A)$ into $(B, \mathcal{N}_B)$ is said to be

(a) *relatively fuzzy minimal continuous* (briefly, (rfm)-continuous), if $f^{-1}(W) \wedge A \in \mathcal{M}_A$ for every fuzzy set W in $\mathcal{N}_B$,

(b) *relatively fuzzy minimal open* (briefly, (rfm)-open), if $f(V) \in \mathcal{N}_B$ for every fuzzy set V in $\mathcal{M}_A$.

Theorem 4.5. *Suppose* $(A, \mathcal{M}_A)$ *and* $(B, \mathcal{N}_B)$ *are fuzzy minimal subspaces of fuzzy minimal spaces* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *respectively. If* $f : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ *is fuzzy* m*-continuous with* $f(A) \leq B$*, then* $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ *is* (rfm)*-continuous.*

Proof. For any given fuzzy m-open set $W \in \mathcal{N}_B$, there exists $\mu \in \mathcal{N}$ for which $W = \mu \wedge B$. Since $f(A) \leq B$, then $A \leq f^{-1}(f(A)) \leq f^{-1}(B)$ and hence

$$\begin{aligned} f^{-1}(W) \wedge A &= f^{-1}(\mu \wedge B) \wedge A \\ &= f^{-1}(\mu) \wedge f^{-1}(B) \wedge A \\ &= f^{-1}(\mu) \wedge A. \end{aligned}$$

Since f is fuzzy m-continuous, so $f^{-1}(\mu) \in \mathcal{M}$, which implies that $f^{-1}(W) \wedge A \in \mathcal{M}_A$. Therefore, $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ is (rfm)-continuous.

The following example shows that the converse of Theorem 4.5 does not hold.

Example 4.6. Suppose $X = \{a, b\}$, $\mathcal{M} = \{\alpha 1_X : \alpha \in I\}$ and $\mathcal{N} = \{\alpha 1_X : \alpha \in I\} \cup b_1$. Let $A = B = a_1$. Since $\mathcal{N} \nsubseteq \mathcal{M}$, so it follows from part (b) of Theorem 3.6 that the identity map $id_X : (X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ is not fuzzy m-continuous. Clearly, $\mathcal{M}_A = \mathcal{N}_B = \{a_\alpha : \alpha \in [0, 1]\}$ where $a_0 = 01_X$. Also, $id_X^{-1}(a_\alpha) \wedge a_1 = a_\alpha \wedge a_1 = a_\alpha \in \mathcal{M}_A$ and so $id_X : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ is (rfm)-continuous.

Theorem 4.7. *The composition of two (rfm)-continuous functions is (rfm)-continuous too.*

Proof. Let $(A, \mathcal{M}_A)$, $(B, \mathcal{N}_B)$ and $(C, \mathcal{Q}_C)$ be fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{Q})$ respectively. Suppose $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ and $g : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ are (rfm)-continuous. We must prove that $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-continuous. To see this, suppose W is an arbitrary element of $\mathcal{Q}_C$. Then $g^{-1}(W) \wedge B \in \mathcal{N}_B$ and so $f^{-1}(g^{-1}(W) \wedge B)) \wedge A \in \mathcal{M}_A$, i.e., $(gof)^{-1}(W) \wedge A \in \mathcal{M}_A$. Therefore, $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-continuous.

Similarly, one can deduce the following result.

Theorem 4.8. *Let $(A, \mathcal{M}_A)$, $(B, \mathcal{N}_B)$ and $(C, \mathcal{Q}_C)$ be fuzzy minimal subspaces of fuzzy minimal spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{Q})$ respectively. Suppose $f : (A, \mathcal{M}_A) \longrightarrow (B, \mathcal{N}_B)$ and $g : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ are (rfm)-open. Then $gof : (A, \mathcal{M}_A) \longrightarrow (C, \mathcal{Q}_C)$ is (rfm)-open.*

Definition 4.9. [14] For each $j \in \{1, 2, ..., n\}$, let A_j be a fuzzy set in X_j. The *fuzzy product* $A = \prod_{j=1}^{n} A_j$ as a fuzzy set of $X = \prod_{j=1}^{n} X_i$ is defined by

$$A(x_1, x_2, ..., x_n) = \min\{A_1(x_1), A_2(x_2), ..., A_n(x_n)\}.$$

We use $A_1 \times A_2 \times \cdots \times A_n$ instead of $\prod_{j=1}^{n} A_j$ and, especially, $A_1 \times A_2$ instead of $\prod_{j=1}^{2} A_j$.

Lemma 4.10. *Suppose A_j is a fuzzy set in X_j for each $j \in \{1, 2, ..., n\}$ and A is the corresponding fuzzy product. Then $\pi_j(A) \leq A_j$ for all $j \in \{1, 2, ..., n\}$.*

Theorem 4.11. *Suppose* $\{(X_j,\mathcal{M}_j) : j \in \{1,...,n\}\}$ *is a family of fuzzy minimal spaces,* $(X,\mathcal{M})$ *is the corresponding fuzzy product minimal space,* A_i *is a fuzzy set in* X_j *for each* $j \in \{1,...,n\}$ *and* $A = \prod_{j=1}^{n} A_j$. *Let* $(B,\mathcal{N}_B)$ *be a fuzzy minimal subspace of the fuzzy minimal space* $(Y,\mathcal{N})$. *Then* $f : (B,\mathcal{N}_B) \longrightarrow (A,\mathcal{M}_A)$ *is* (rfm)*-continuous if and only if* $\pi_j of : (B,\mathcal{N}_B) \longrightarrow (A_j,\mathcal{M}_{jA_j})$ *is* (rfm)*-continuous for all* $j \in \{1,...,n\}$.

Proof. One direction is an immediate consequence of Theorem 4.5 and Theorem 4.7. For the converse, on the contrary suppose $\pi_i of$ is (rfm)-continuous for each $i \in \{1,...,n\}$ and f is not (rfm)-continuous. Hence, there exists $V \in \mathcal{M}_A$ such that $f^{-1}(V) \wedge B \notin \mathcal{N}_B$ and now by Definition 4.1 there exists $U \in \mathcal{M}$ such that $f^{-1}(U \wedge A) \wedge B \notin \mathcal{N}_B$. According to (3.3) there exist $l \in \{1,...,n\}$ and $U_l \in \mathcal{M}_l$ for which

$$f^{-1}(1_{X_1} \times \cdots \times 1_{X_{l-1}} \times U_l \times 1_{X_{l+1}} \times \cdots \times 1_{X_n}) \wedge f^{-1}(A) \wedge B \notin \mathcal{N}_B.$$

Then, Lemma 4.10 and the fact that $B \leq f^{-1}(A) \leq (\pi_l of)^{-1}(A_l)$ imply

$$(\pi_l of)^{-1}(U_l \wedge A_l) \wedge B \notin \mathcal{N}_B;$$

i.e., $\pi_l of$ is not (rfm)-continuous, which is a contradiction.

Corollary 4.12. *Suppose* $(X,\mathcal{M})$ *is a minimal space and* $\{(Y_j,\mathcal{N}_j) : j \in \{1,...,n\}\}$ *is a finite family of fuzzy minimal spaces and* $(Y,\mathcal{N})$ *is their corresponding fuzzy product minimal spaces. Also, suppose* A *and* B_j *are respectively fuzzy sets in* X *and* Y_j *for each* $j \in \{1,...,n\}$ *and* $B = \prod_{i=1}^{n} B_i$. *Let* f_j *be a mapping of* $(A,\mathcal{M}_A)$ *to* $(B_j,\mathcal{N}_{jB_j})$. *Then* $f : (A,\mathcal{M}_A) \longrightarrow (B,\mathcal{N}_B)$ *defined by* $f(x) = (f_1(x)...f_n(x))$ *is* (rfm)*-continuous if and only if* $f_j : (A,\mathcal{M}_A) \longrightarrow (B_j,\mathcal{N}_{jB_j})$ *is* (rfm)*-continuous for each* $j = 1,...,n$.

Proof. It follows from Theorem 4.11 and the fact that $\pi_j of = f_j$.

Theorem 4.13. *Suppose* $(X,\mathcal{M})$, $(Y,\mathcal{N})$ *are fuzzy minimal spaces,* $C = A \times B$, $\mathcal{Q} = \mathcal{M} \times \mathcal{N}$ *and also* A *and* B *are fuzzy sets in* X *and* Y *respectively. Then for each* $y_0 \in Y$ *with* $B(y_0) \geq A(x)$ *for all* $x \in X$, *the mapping* $i_{y_o} : (A,\mathcal{M}_A) \longrightarrow (C,\mathcal{Q}_C)$ *defined by* $i_{y_0}(x) = (x,y_0)$ *is* (rfm)*-continuous.*

Proof. First, we show that $i_{y_0}(A) \leq C$. It is easy to see that

$$i_{y_0}(A)(x,y) = \begin{cases} A(x) & y = y_0 \\ 0 & \text{otherwise} \end{cases}.$$

Since for each $y_0 \in Y$ with $B(y_0) \geq A(x)$ for all $x \in X$, so one can deduce that $i_{y_0}(A) \leq C$. That i_{y_0} is (rfm)-continuous follows from Theorem 3.11 and Theorem 4.5.

Similarly, using Theorem 3.12 and Theorem 4.5 one can deduce the following result.

Theorem 4.14. *Suppose* $(X,\mathcal{M})$, $(Y,\mathcal{N})$ *are fuzzy minimal spaces,* $C = A \times B$, $\mathcal{Q} = \mathcal{M} \times \mathcal{N}$ *and also* A *and* B *are fuzzy sets in* X *and* Y *respectively. Then for each* $x_0 \in X$ *with* $A(x_0) \geq B(y)$ *for all* $y \in Y$*, the mapping* $i_{x_o} : (B, \mathcal{N}_B) \longrightarrow (C, \mathcal{Q}_C)$ *defined by* $i_{x_0}(y) = (x_0, y)$ *is* (rfm)*-continuous.*

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