COMMON FIXED POINT FOR LIPSCHITZIAN MAPPING SATISFYING RATIONAL CONTRACTIVE CONDITIONS

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Abstract. Common fixed point theorems for a class of mappings called occasionally weakly compatible in a symmetric space (X, d) under Lipschitzian type rational contractive conditions are obtained.

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1. Introduction and preliminaries

In 1968, Kannan [14] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades (see, for example, [11] for a listing and comparison of many of these definitions). A number of these papers dealt with fixed points for more than one map. In some cases commutativity between the maps was required in order to obtain a common fixed point. Sessa [13] coined the term weakly commuting. Jungck [8] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps [9]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. Also, during this time a number of authors established fixed point theorems for pair of maps (see, for example, [5], [12]). Recently, Thagafi and Shahzad [4] gave the definition which is a proper generalization of nontrivial weakly compatible maps which do have coincidence points (see, also, [2] and [10]). The aim of this paper is to obtain some fixed points theorem involving occasionally weakly compatible maps in the setting of symmetric space satisfying a rational contractive condition. Our results complement, extend and unify several well known comparable results.

Definition 1.1. Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

The following concept is a proper generalization of nontrivial weakly compatible maps which a coincidence point.

Definition 1.2. Two selfmaps f and g of a set X are said to be occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Our theorems are proved in symmetric spaces which are more general than metric spaces.

Definition 1.3. Let X be a set. A symmetric on X is a mapping $d : X \times X \to (0, \infty)$ such that

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

and

$$d(x, y) = d(y, x) \quad \text{for} \quad x, y \in X.$$

We shall also need the following Proposition from [10] (see also [1]).

Proposition 1.4. *Let f and g be occasionally weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2. Common fixed point theorems

The following result generalizes Theorem 4 of [7].

Theorem 2.1. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$d(Ax, By) \leq a \left[\frac{(d(Ax, Sx))^2 + (d(By, Ty))^2}{d(Ax, Sx) + d(By, Ty)} \right] + bd(Sx, Ty) \tag{2.1}$$

if $d(Ax, Sx) + d(By, Ty) \neq 0$ or $d(Ax, By) = 0$, if $d(Ax, Sx) + d(By, Ty) = 0$, where $a, b > 0$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are each owc, there exist points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. From (2.1), we have $Ax = By$. Therefore $Ax = Sx = By = Ty$. Moreover, if there is another point z such that $Az = Sz$, then, using (2.1) it follows that $Az = Sz = By = Ty$, or $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Proposition 1.4, w is the only common fixed point of A and S. By symmetry there is a unique point $z \in X$ such that $z = Bz = Tz$. From (2.1), we obtain, $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique.

Theorem 1 of Ahmad and Imdad [3] is a special case of Theorem 2.1.

Theorem 2.2. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$(2.2)\quad d(Ax, By) \leq a\left[\frac{d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ty, Ax)}{d(Sx, By) + d(Ty, Ax)}\right] + bd(Sx, Ty)$$

if $d(Sx, By) + d(Ty, Ax) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, By) + d(Ty, Ax) = 0$, where $a, b > 0$ and $b < 1$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. Since the pairs $\{A, S\}$ and $\{B, T\}$ are each owc, there exist points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. Now we claim that $Ax = By$, if not, then $d(Sx, By) + d(Ty, Ax) \neq 0$. From (2.2),

$$d(Ax, By) \leq bd(Ax, By),$$

a contradiction. Thus, we have $Ax = By$, and $Ax = Sx = By = Ty$. Moreover, if there is another point z such that $Az = Sz$, then, $Az = By$. If not, then $d(Sz, By) + d(Ty, Az) \neq 0$. Using (2.2),

$$d(Az, By) \leq bd(Az, By),$$

a contradiction and hence, it follows that $Az = Sz = By = Ty$, or $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Proposition 1.4, w is the only common fixed point of A and S. By symmetry there is a unique point $z \in X$ such that $z = Bz = Tz$. From (2.2), we obtain, $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique.

Theorem 5 of [7] and Theorem 2 of Ahmad and Imdad [6] are a special cases of Theorem 2.2.

Theorem 2.3. *Let A, B, S and T be self mappings of a symmetric space X with symmetric d, and*

$$(2.3)\quad d(Ax, By) \leq \frac{ad(Ax, Sx)d(Ty, By) + bd(Sx, By)d(Ty, Ax)}{d(Sx, Ax) + d(By, Ty)}] + cd(Sx, Ty)$$

if $d(Sx, Ax) + d(By, Ty) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, Ax) + d(By, Ty) = 0$, where $a, b, c > 0$. Then A, B, S and T have a unique common fixed point if the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible.

Proof. The proof is similar to that of Theorem 2.1, and will therefore be omitted.

Theorem 2.4. *Let A, B,and S be self mappings of a symmetric space X with symmetric d, and*

$$(2.4)\quad \begin{aligned} d(Ax, By) &\leq \left[\frac{ad(Ax, Sx)d(Sy, By) + bd(Sx, By)d(Sy, Ax)}{d(Sx, Ax) + d(By, Sy)}\right] \\ &+ c\left[\frac{d(Ax, Sx)d(Sy, Ax) + d(By, Sy)d(Sx, By)}{d(Sx, Ax) + d(By, Sy)}\right] \end{aligned}$$

if $d(Sx, Ax) + d(By, Sy) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, Ax) + d(By, Sy) = 0$, where $a, b, c > 0$ and $c < 1$. Then A, B and T have a unique common fixed point if one of the pairs $\{A, B\}$ or $\{A, S\}$ is occasionally weakly compatible.

Proof. Suppose that $\{A, S\}$ is owc. Then there exists a point $x \in X$ such that $Ax = Sx$. Now we claim that $Sx = Bx$. If not, then

$$d(Ax, Bx) \leq c\,\frac{(d(Bx, Sx))^2}{d(Bx, Sx)} = cd(Ax, Bx),$$

a contradiction. Thus, $Ax = Bx = Sx$. Now, $AAx = ASx = SAx = SSx$. Since $Bx = Sx$ and $SSx = ASx$, therefore, $d(SSx, ASx) + d(Bx, Sx) = 0$, and $d(ASx, Bx) = 0$. Hence, $ASx = Bx$, which shows Sx is a fixed point of A. Also, Sx is a fixed point of S. Suppose that $Sx \neq BSx$. From (2.4), we have

$$d(Ax, BSx) \leq c\,\frac{(d(BSx, Sx))^2}{d(BSx, Sx)} = cd(BSx, Sx)$$

a contradiction. Therefore, Sx is a common fixed point of A, B and S. Let w and z be two common fixed point of A, B and S. Since

$$d(Sw, Aw) + d(Bz, Sz) = 0,$$

therefore $d(Aw, Bz) = d(w, z) = 0$, $w = z$. A similar argument applies if the pair $\{A, B\}$ is owc.

Theorem 2.5. *Let A, B, and S be self mappings of a symmetric space X with symmetric d, and*

$$(2.5)\quad \begin{aligned} d(Ax, By) &\leq a\,\frac{d(Sx, By)d(Sx, Sy)}{d(Sx, Sy) + d(Sy, By)}] + b[d(Sx, Ax) + d(Sy, By)] \\ &\quad c[d(Sx, By) + d(Sy, Ax)] + dd(Sx, Sy) \end{aligned}$$

if $d(Sx, Sy) + d(Sy, By) \neq 0$ or $d(Ax, By) = 0$, if $d(Sx, Sy) + d(Sy, By) = 0$, where $a, b, c > 0$ and $a + b + 2c + d < 1$. Then A, B and T have a unique common fixed point if one of the pairs $\{A, B\}$ or $\{A, S\}$ is occasionally weakly compatible.

Proof. Suppose that $\{A, S\}$ is owc. Then there exist a point $x \in X$ such that $Ax = Sx$. Now we claim that $Sx = Bx$. If not,

$$d(Ax, Bx) \leq bd(Sx, Bx) + cd(Sx, Bx)$$

a contradiction. Thus, $Ax = Bx = Sx$. Now, $AAx = ASx = SAx = SSx$. Since $Bx = Sx$ and $SSx = ASx$, therefore, $d(SSx, ASx) + d(Bx, Sx) = 0$, and $d(ASx, Bx) = 0$. Hence, $ASx = Bx$, which shows Sx is a fixed point of A. Also, Sx is a fixed point of S. Suppose that $Sx \neq BSx$. From (2.5), we have

$$\begin{aligned} d(Ax, BSx) &\leq a\,\frac{d(Sx, BSx)d(Sx, SSx)}{d(SSx, BSx)} \\ &= a\,\frac{d(Sx, BSx)d(Sx, Sx)}{d(Sx, BSx)} = 0 \end{aligned}$$

a contradiction. Therefore, Sx is a common fixed point of A, B and S. Let w and z be two common fixed point of A, B and S. If, $w \neq z$, then,

$$\begin{aligned} d(w, z) &= d(Aw, Bz) \\ &\leq ad(Sw, Bz) + c[d(Sw, Bz) + d(Sz, Aw)] + dd(Sw, Sz) \\ &= (a + 2c + d)d(w, z) \end{aligned}$$

a contradiction, therefore $w = z$. The proof of the result assuming $\{A, B\}$ is owc is similar.

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