CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING SĂLĂGEAN OPERATOR

J.K. Prajapat

Department of Mathematics
Central University of Rajasthan
Kishangarh-305802, Distt.-Ajmer Rajasthen
India
e-mail: jkp_0007@rediffmail.com

R.K. Raina

10/11 Ganpati Vihar, Opposite Sector 5
Udaipur 313002, Rajasthan
India
e-mail: rainark_7@hotmail.com

Abstract. The familiar Sălăgean operator is used here to define a new subclass of analytic and univalent functions in the open unit disk $\mathbb{U}$. In this note we obtain some sufficient conditions for functions belonging to this class and mention few important consequences of our main results.

2000 Mathematics Subject Classification: 30C45.

Keywords and phrases: Starlike functions; Convex functions; Close-to-Convex functions; Strongly starlike functions; Strongly convex functions; Sălăgean operator; Jack's Lemma; Argument properties.

1. Introduction, definitions and key lemmas

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also, the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{P}(\alpha)$ defined in the open unit $\mathbb{U}$ are the well known subclasses of the class $\mathcal{A}$ of order $\alpha(0 \leqq \alpha < 1)$ in $\mathbb{U}$ which have been studied quite extensively in the *Geometric Function Theory*, and one may refer to MacGregor [6] and Srivastava and Owa ([11], [12]) for their various details.

Let $\mathcal{S}^*(\alpha_1, \alpha_2)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\left(\frac{zf'(z)}{f(z)}\right) < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U};\ 0 < \alpha_1;\ \alpha_2 \leqq 1) \tag{1.2}$$

and let $\mathcal{K}(\alpha_1, \alpha_2)$ be the subclass of $\mathcal{A}$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U};\ 0<\alpha_1;\ \alpha_2 \leqq 1), \tag{1.3}$$

where $\mathcal{S}^*(\alpha_1,\alpha_2)$ and $\mathcal{K}(\alpha_1,\alpha_2)$ are the subclasses of $\mathcal{A}$ introduced and studied by Takahashi and Nunokawa [13].

We observe that

$$\mathcal{S}^*(\alpha,\alpha) = \mathcal{S}^*_s(\alpha) \qquad \text{and} \qquad \mathcal{K}(\alpha,\alpha) = \mathcal{K}_c(\alpha),$$

where $\mathcal{S}^*_s(\alpha)$ and $\mathcal{K}_c(\alpha)$, are respectively, the familiar subclasses of $\mathcal{A}$ consisting of functions which are strongly starlike of order α $(0<\alpha\le 1)$ in $\mathbb{U}$ and strongly convex of order α $(0<\alpha\le 1)$ in $\mathbb{U}$. Also, we note that $\mathcal{S}^*_s(0)=\mathcal{S}^*$ and $\mathcal{K}_c(0)=\mathcal{K}$ (see, for details [11] and [12]).

For $f(z)\in\mathcal{A}$, Sălăgean [10] introduced a derivative operator $\mathcal{D}^n$ of order n which we define here by

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n\, a_k z^k \qquad (f\in\mathcal{A}; n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}). \tag{1.4}$$

In terms of the Sălăgean operator $\mathcal{D}^\lambda$ $(\lambda\in\mathbb{N}_0)$ defined by (1.4) above, we introduce a new subclass of $\mathcal{A}$ denoted by $\mathcal{B}(\lambda,\mu,\alpha)$ consisting of functions of the form (1.1) which satisfy the following inequality:

$$\left|\frac{z^{\mu-2}\,\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^\lambda f(z))^{\mu-1}} - 1\right| < 1-\alpha. \tag{1.5}$$

We observe that on specializing the arbitrary parameters λ and μ, the above class $\mathcal{B}(\lambda,\mu,\alpha)$ yields the following:

$$\mathcal{B}(0,2,\alpha)=\mathcal{S}^*(\alpha);\ \mathcal{B}(1,2,\alpha)=\mathcal{K}(\alpha);\ \mathcal{B}(0,1,\alpha)=\mathcal{P}(\alpha) \text{ and } \mathcal{B}(0,3,\alpha)=\mathcal{B}(\alpha), \tag{1.6}$$

where $\mathcal{B}(\alpha)$ is a subclass of $\mathcal{A}$ which was earlier studied by Frasin and Darus [2] (see also [1]).

In order to derive our main results, we recall the following known lemmas.

Lemma 1. (Jack's Lemma [4]) *Let the nonconstant function $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, then*

$$z_0 w'(z_0) = \gamma w(z_0). \tag{1.7}$$

where γ is real and $\gamma\geqq 1$.

Lemma 2. ([7]) *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\phi(z)$ is a mapping from $\mathbb{C}^2\times\mathbb{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z)\notin\Omega$ for $z\in\mathbb{U}$, and for all real x,y such that $y\leqq -(1+x^2)/2$. If the function $q(z)=1+q_1z+q_2z^2+\cdots$ is analytic in $\mathbb{U}$ such that $\phi(q(z),zq'(z);z)\in\Omega$ for all $z\in\mathbb{U}$, then $\Re(q(z))>0$.*

Lemma 3. ([8]) *Let a function $q(z)$ be analytic in $\mathbb{U}$ with $q(0)=1$ and $q(z)\neq 0$ $(z\in\mathbb{U})$. If there exists two points $z_1,\ z_2\in\mathbb{U}$ such that*

$$-\frac{\pi\alpha_1}{2}=\arg\big(q(z_1)\big)<\arg\big(q(z)\big)<\arg\big(q(z_2)\big)=\frac{\pi\alpha_2}{2} \tag{1.8}$$

for $\alpha_1>0,\ \alpha_2>0$ and for all $|z|<|z_1|=|z_2|$, then

$$\frac{z_1q'(z_1)}{q(z_1)}=-i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \qquad \text{and} \qquad \frac{z_2q'(z_2)}{q(z_2)}=i\left(\frac{\alpha_1+\alpha_2}{2}\right)m, \tag{1.9}$$

where

$$m\geqq\frac{1-|a|}{1+|a|} \qquad \text{and} \qquad a=i\ tan\ \frac{\pi}{2}\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right). \tag{1.10}$$

In this note we investigate sufficient conditions for functions in the class $\mathcal{A}$ to be members of the class $\mathcal{B}(\lambda,\mu,\alpha)$ (which is defined by involving the familiar Sălăgean operator). Some corollaries are deduced exhibiting the usefulness of the main results.

2. A set of sufficient conditions

Making use of Lemma 1, we first prove

Theorem 1. *If $f(z)\in\mathcal{A}$ satisfies the following inequality:*

$$\left|\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)-1\right|<\frac{1-\alpha}{2-\alpha} \tag{2.1}$$

$$(z\in\mathbb{U};\ 0\leqq\alpha<1;\ \lambda\in\mathbb{N}_0;\ \mu\geqq 0),$$

then $f(z)\in\mathcal{B}(\lambda,\mu,\alpha)$.

Proof. Let $f(z)\in\mathcal{A}$. Define a function $w(z)$ by

$$\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}=1+(1-\alpha)w(z), \tag{2.2}$$

then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. If follows from (2.2) that

$$\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)-1=\frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}. \tag{2.3}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)|=|w(z_0)|=1,$$

then (2.3) in view of (1.7) of Lemma 1 and $w(z_0) = e^{i\theta}(0 \leq \theta < 2\pi)$ yields

$$\begin{aligned}(2.4)\qquad &\left|\frac{\mathcal{D}^{\lambda+2}f(z_0)}{\mathcal{D}^{\lambda+1}f(z_0)} - (\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z_0)}{\mathcal{D}^{\lambda}f(z_0)} - 1\right) - 1\right| \\ &= \left|\frac{(1-\alpha)z_0w'(z_0)}{1+(1-\alpha)w(z_0)}\right| = \frac{|(1-\alpha)\gamma e^{i\theta}|}{|1+(1-\alpha)e^{i\theta}|} \geqq \frac{1-\alpha}{2-\alpha}\end{aligned}$$

which contradicts (2.1).

Therefore $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$ and consequently (2.2) gives

$$(2.5)\qquad \left|\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}} - 1\right| = |(1-\alpha)w(z)| < 1-\alpha \qquad (z \in \mathbb{U})$$

which implies that $f(z) \in \mathcal{B}(\lambda, \mu, \alpha)$, completing the proof of Theorem 1.

Remark 1. In view of the relationships that $\mathcal{B}(0, 2, \alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{B}(1, 2, \alpha) = \mathcal{K}(\alpha)$ and $\mathcal{B}(0, 1, \alpha) = \mathcal{P}(\alpha)$, Theorem 1 would easily lead to the results giving sufficient conditions for the function $f(z)$ defined by (1.1) to belong, respectively, to the subclasses $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{P}(\alpha)$. The special cases corresponding to the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are also identifiable with the results due to Irmak *et al.* [3, p. 364]. Also, we note that by setting $\lambda = 0$ and $\mu = 3$, Theorem 1 corresponds to the result of Frasin and Darus [2, p. 307, Theorem 2.4].

Next we prove

Theorem 2. *If $f(z) \in \mathcal{A}$ satisfies the following inequality:*

$$\begin{aligned}(2.6)\qquad &\Re\left\{\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}\left[\alpha\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}} + \alpha\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}\right.\right. \\ &\left.\left. -\alpha(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} - 1\right) + (1-2\alpha)\right]\right\} > \alpha\beta\left(\beta - \frac{1}{2}\right) - \frac{\alpha}{2} + \beta \\ &\qquad (z \in \mathbb{U};\ \mu \geqq 0;\ \lambda \in \mathbb{N}_0;\ \alpha \geqq 0;\ 0 \leqq \beta < 1),\end{aligned}$$

then $f(z) \in \mathcal{B}(\lambda, \mu, \beta)$.

Proof. Define a function $q(z)$ by

$$(2.7)\qquad \frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} = \beta + (1-\beta)q(z),$$

then $q(z)$ is of the form $q(z) = 1+q_1z+q_2z^2+...$ and is analytic in $\mathbb{U}$. Differentiating both sides of (2.7) with respect to z, we get

$$(2.8)\qquad \frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)} - (\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)} - 1\right) - 1 = \frac{(1-\beta)zq'(z)}{\beta+(1-\beta)q(z)}.$$

Using (2.7) and (2.8), we obtain

$$\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}\left[\alpha\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(\mathcal{D}^{\lambda}f(z))^{\mu-1}}+\alpha\frac{\mathcal{D}^{\lambda+2}f(z)}{\mathcal{D}^{\lambda+1}f(z)}-\alpha(\mu-1)\left(\frac{\mathcal{D}^{\lambda+1}f(z)}{\mathcal{D}^{\lambda}f(z)}-1\right)+(1-2\alpha)\right]$$
$$=\alpha(1-\beta)zq'(z)+\alpha[\beta+(1-\beta)q(z)]^2+(1-\alpha)[\beta+(1-\beta)q(z)]$$
$$=\alpha(1-\beta)zq'(z)+\alpha(1-\beta)^2q^2(z)+(1-\beta)(1+2\alpha\beta-\alpha)q(z)+\beta(\alpha\beta+1-\alpha)$$
$$=\phi(q(z),zq'(z);z),$$

where

$$\phi(r,s;t)=\alpha(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)(1+2\alpha\beta-\alpha)r+\beta(\alpha\beta+1-\alpha). \tag{2.9}$$

For all real values of x and y satisfying $y \leqq -(1+x^2)/2$, we infer that

$$\begin{aligned}\Re(\phi(ix,y;z)) &= \alpha(1-\beta)y-\alpha(1-\beta)^2x^2+\beta(\alpha\beta+1-\alpha)\\ &\leqq -\frac{\alpha}{2}(1-\beta)-\left[\frac{\alpha}{2}(1-\beta)+\alpha(1-\beta)^2\right]x^2+\beta(\alpha\beta+1-\alpha)\\ &\leqq \beta(\alpha\beta+1-\alpha)-\frac{\alpha}{2}(1-\beta).\end{aligned}$$

Let $\Omega=\left\{w:\ \Re(w)>\alpha\beta\left(\beta-\frac{1}{2}\right)-\frac{\alpha}{2}+\beta\right\}$, then $\phi(q(z),zq'(z);z)\in\Omega$ and $\phi(ix,y;z)\notin\Omega$ for all real x and $y\leqq -(1+x^2)/2$, $z\in\mathbb{U}$. Applying Lemma 2 we conclude that $\Re(q(z))>0$, which in view of (1.5) and (2.7) implies that $f(z)\in\mathcal{B}(\lambda,\mu,\beta)$.

If we set $\mu=3$ and $\lambda=0$, then Theorem 2 gives the following result.

Corollary 1. *If $f(z)\in\mathcal{A}$ satisfies the inequality*

$$\begin{aligned}&\Re\left\{\frac{z^2f(z)}{(f(z))^2}\left[\alpha\left(\frac{z^2f(z)}{(f(z))^2}+\frac{zf''(z)}{f'(z)}-\frac{2zf'(z)}{f(z)}+1\right)+1\right]\right\}\\ &\qquad>\alpha\beta\left(\beta-\frac{1}{2}\right)-\frac{\alpha}{2}+\beta \qquad (z\in\mathbb{U};\ \alpha\geqq 0;\ 0\leqq\beta<1),\end{aligned} \tag{2.10}$$

then $f(z)\in\mathcal{B}(\beta)$.

Remark 2. Upon making similar parametric substitutions as pointed out in Remark 1 above, interesting sufficient conditions can be obtained for the subclasses $\mathcal{S}^*(\beta)$, $\mathcal{K}(\beta)$ and $\mathcal{P}(\beta)$ which would evidently include a known result involving the subclass $\mathcal{S}^*$ of starlike functions due to Li and Owa [5] (see also [9, p.3, Corollary 2.2]). Derivation of these special cases being straightforward, we skip mentioning of these results.

3. Argument properties

Making use of Lemma 3, we prove

Theorem 3. *Let*

$$\frac{z^{\mu-2}\mathcal{D}^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} \neq \beta \qquad (z \in \mathbb{U};\ \mu \geqq 0;\ \lambda \in \mathbb{N}_0;\ 0 \leqq \beta < 1).$$

If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\begin{aligned} &-\frac{\pi\alpha_1}{2} - tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\right) \\ &< arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}}\left[\frac{D^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - (\mu-1)\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\right) + \frac{\gamma+\beta-1}{1-\beta}\right] - \frac{\gamma\beta}{1-\beta}\right\} \\ &< \frac{\pi\alpha_2}{2} + tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\right) \qquad (0 < \alpha_1;\ \alpha_2 \leqq 1;\ \gamma > 0), \end{aligned} \tag{3.1}$$

then

$$-\frac{\pi\alpha_1}{2} < arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}} - \beta\right\} < \frac{\pi\alpha_2}{2}. \tag{3.2}$$

Proof. Let $q(z)$ be the same function as defined in (2.7), then since $q(z)$ is analytic in the open unit disk $\mathbb{U}$ with $q(0) = 1$, it follows from the hypothesis of Theorem 3 that $q(z) \neq 0$. Following (2.8), we obtain

$$\begin{aligned} &\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^{\lambda}f(z))^{\mu-1}}\left[\frac{D^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - (\mu-1)\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\right)\right. \\ &\qquad \left. + \frac{\gamma+\beta-1}{1-\beta}\right] - \frac{\gamma\beta}{1-\beta} = (1-\beta)zq'(z) + \gamma q(z). \end{aligned} \tag{3.3}$$

Suppose now that there exists two points $z_1,\ z_2 \in \mathbb{U}$ such that the conditions (1.8) are satisfied. Applying (1.9) and (1.10) of Lemma 3, we get

$$\begin{aligned} &\arg\big(\gamma q(z_1) + (1-\beta)z_1q'(z)\big) = \arg(q(z_1)) + \arg\left(\gamma + (1-\beta)\frac{z_1q'(z_1)}{q(z_1)}\right) \\ &= -\frac{\pi\alpha_1}{2} + \arg\left(\gamma - i\frac{(\alpha_1+\alpha_2)(1-\beta)}{2}\, m\right) = -\frac{\pi\alpha_1}{2} - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\, m\right) \\ &\leqq -\frac{\pi\alpha_1}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\, m\right), \end{aligned}$$

which, by virtue of (3.3), contradicts the assumption stated in (3.1). Similarly, we can show that

$$\arg\big(\gamma q(z_2) + (1-\beta)z_2q'(z_2)\big) \geqq \frac{\pi\alpha_1}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(1-\beta)}{2\gamma}\, m\right),$$

which again contradicts the assumption mentioned in (3.1). Hence the function $q(z)$ defined by (2.7) satisfies the inequality

$$-\frac{\pi\alpha_1}{2} < \arg(q(z)) < \frac{\pi\alpha_2}{2},$$

which implies that

$$-\frac{\pi\alpha_1}{2} < \arg\left\{\frac{z^{\mu-2}D^{\lambda+1}f(z)}{(D^\lambda f(z))^{\mu-1}} - \beta\right\} < \frac{\pi\alpha_2}{2} \qquad (z \in \mathbb{U}).$$

This completes the proof of Theorem 3.

If we set $\alpha_1 = \alpha_2 = \alpha, \mu = 2$ and $\lambda = \beta = 0$ in Theorem 3, we get Corollary 2 below.

Corollary 2. *Let*

$$\frac{zf'(z)}{f(z)} \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\left|arg\left\{\frac{zf'(z)}{f(z)}\left(1+\gamma+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\pi\alpha}{2} + tan^{-1}\frac{\alpha}{\gamma} \qquad (0 < \alpha \leqq 1;\ \gamma > 0), \tag{3.4}$$

then $f(z) \in S_s^*(\alpha)$.

Also, if we put $\alpha_1 = \alpha_2 = \alpha, \mu = 2, \lambda = 1$ and $\beta = 0$ in Theorem 3, we get

Corollary 3. *Let*

$$\left(1+\frac{zf''(z)}{f'(z)}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\left|arg\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{z(zf'''(z)+2f''(z))}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}+\gamma\right)\right\}\right| < \frac{\pi\alpha}{2} + tan^{-1}\frac{\alpha}{\gamma} \qquad (0 < \alpha \leqq 1;\ \gamma > 0), \tag{3.5}$$

then $f(z) \in \mathcal{K}_c^*(\alpha)$.

Lastly, by choosing $\alpha_1 = \alpha_2 = 1, \mu = 1$ and $\lambda = 0$ in Theorem 3, we obtain

Corollary 4. *Let*

$$f'(z) \neq \beta \qquad (z \in \mathbb{U};\ 0 \leqq \beta < 1).$$

If $f(z) \in \mathcal{A}$ satisfies the inequality

$$\left|arg\left\{f'(z)\left(\frac{zf''(z)}{f'(z)}+\frac{\gamma}{1-\beta}\right)-\frac{\gamma\beta}{1-\beta}\right\}\right| < \frac{\pi}{2} + tan^{-1}\left(\frac{1-\beta}{\gamma}\right) \quad (\gamma > 0), \tag{3.6}$$

then $f(z) \in \mathcal{P}(\beta)$.

Acknowledgements. The authors express their sincerest thanks to the referee for useful suggestions.

References

[1] Frasin, B.A., *A note on certain analytic and univalent functions*, South East Asian J. Math., 28 (2004), 829-836.

[2] Frasin, B.A. and Darus, M., *On certain analytic univalent functions*, Int. J. Math. and Math. Sci., 25 (5) (2001), 305-310.

[3] Irmak, H., Raina, R.K. and Owa, S., *Certain results involving inequalities of analytic and univalent functions,* Far East J. Math. Sci., 10 (2003), 359-366.

[4] Jack, I.S., *Functions starlike and convex of order* α*,* J. Londan Math. Soc., 2 (3) (1971), 469-474.

[5] Li, J.L. and Owa, S., *Sufficient conditions for starlikeness*, *Indian J. Pure Appld. Math.*, 33 (2002), 313-318.

[6] MacGregor, T.H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., 104 (1962), 532-537.

[7] Miller, S.S. and Mocanu, P.T., *Differential subordinates and inequalities in the complex plane*, J. Differ. Equations, 67 (1987), 199-211.

[8] Nunokawa, M., Owa, S., Saitoh, H., Cho, N.E. and Takahashi, N., *Some properties of analytic functions at extremal points of arguments*, Preprint.

[9] Ravichandran, V., Selvaraj, C. and Rajalaksmi, R., *Sufficient conditions for starlike functions of order* α, J. Inequal. Pure Appl. Math., 3 (5) (2002), Article 81, 1-6 (electronic).

[10] Sălăgean, G.S., *Subclasses of univalent functions*, In: Complex Analysis, Fifth Romanian Finnish Seminar, Part I (Bucharest, 1981), Lecture Notes in Math., vol. 1013 (1983), Springer, Berlin, 362-372.

[11] Srivastava, H.M. and Owa, S., (Editors), *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[12] Srivastava, H.M. and Owa, S., *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hongkong, 1992.

[13] Takahashi, N. and Nunokawa, M., *A certain connections between starlike and convex functions*, Appl. Math. Lett., 6 (2003), 563-655.

Accepted: 14.10.2008